

A FINITE STRAIN THEORY OF ELASTOPLASTICITY
AND ITS APPLICATION TO WAVE PROPAGATION

by

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for the degree of Doctor of Philosophy

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AND ITS APPLICATION TO WAVE PROPAGATION

T. GÜLTOP

DECLARATION

I, Tekin Gültop declare that, except where reference is made to the work of others, this is my own work. No part of this thesis has been submitted for a degree at another university.

Signed by candidate

T. GÜLTOP

To the Memory of My Mother

and

To My Father

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ABSTRACT

A constitutive theory of finite strain plasticity is developed by using the methods of convex analysis. The theory abstracts and extends the classical assumptions of a convex region of admissible stresses, and the normality law. The overall effects of plastic behaviour are contained in the theory through the presence of one or more internal variables. The thermodynamic restrictions of the second law together with the use of results of convex analysis lead in a natural way to the evolution equation or flow law.

Non-smooth yield surfaces are included in the theory; nevertheless, the form of this theory makes a study of propagation of singular surfaces awkward. With a view to carrying out such a study, an alternative means of treating non-smooth convex yield surfaces is developed. This alternative theory is essentially a synthesis of the theory of Sewell, and that presented earlier in the thesis.

The theory of singular surfaces is reviewed in the context of finite strain elastoplasticity, and necessary conditions for the propagation of acceleration waves are derived. A comparison of elastic and plastic wave speeds is made, and inequalities similar to those of Mandel for the small-strain case are derived. The propagation conditions for principal waves in both longitudinal and transverse directions, and the corresponding wave speeds, are found and compared for solids obeying a neo-Hookean elastic law, and with either the von Mises or Tresca yield criteria.

ÖZET

Konveks analiz metotlari kullanilarak sonlu deformasyonlu plastik ortamlar icin bir bünye teorisi gelistirilmistir. Bu teori, temel varsayimlardan konveks bir gerceklesebilir gerilmeler alanıyla dikeylik yasasina dayanmakta ve bu varsayimlari genisletmektedir. Teoride toplu plastik davranis etkileri bir veya birden fazla ic deęiskenle belirtilmistir. Ikinci termodinamik yasasinin sinirlamalariyla birlikte konveks analiz sonuclarinin kullanimi, gelisme denklemini veya akim yasasini tabii bir sekilde ortaya cikarmistir.

Teori, düzgün olmayan akma yüzeylerini icine almakla birlikte, bu durumuyla tekillik yüzeylerinin yayilmasi konusunun arastirilmasina elverisli deęildir. Böyle bir arastirmayi yürütmek amaciyla, düzgün olmayan konveks akma yüzeylerini incelemek icin farkli bir yol secilmistir. Bu yol aslinda, tezin ilk kisimlarinda sunduęumuz Sewell teorisinin önceki calismamizla bir birlesimidir.

Tekillik yüzeyleri teorisi sonlu deformasyonlu elastoplastik ortam dikkate alinarak gözden gecirilmis ve ivme dalgalarinin yayilmasi icin gerekli sartlar bulunmustur. Elastik ve plastik dalga hizlari karsilastirilmis ve Mandel' in daha önceleri küçük deformasyonlu ortamlar icin bulduęu esitsizliklere ulasilmistir. Esas yönlü, yanali ve boylamasina dalgalar icin yayilma sartlariyla yayilma hizlari hem düzgün, hem de düzgün olmayan akma yüzeyleri icin bulunmustur.

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CHAPTER I

INTRODUCTION

The theory of plasticity in its classical small-strain form is well established, especially that form which the theory takes in its application to metals. The finite-strain theory, on the other hand, while showing indications that it is on the way to becoming an established branch of mechanics, is still nevertheless the subject of considerable effort and debate, and certain of its aspects remain unsettled. The literature on the subject has in the meantime acquired voluminous proportions. In this Chapter we will discuss some of the more important contributions to the finite-strain theory of plasticity, as well as other publications which are particularly relevant to our study.

The propagation of waves, and in particular of acceleration waves, in solid media is another field of continuum mechanics which has been investigated by numerous researchers. A prerequisite for the study of acceleration waves in solid materials is the availability of a sound constitutive theory of the relevant material. A big portion of the contributions to the theory of waves in solid media is in the context of elastic and thermoelastic constitutive assumptions. In this Chapter we will also mention the most relevant contributions in the field of acceleration waves (which are modelled

as propagating singular surfaces) in elastoplastic materials, since the aim of this thesis is to develop a finite-strain theory of plasticity and then to apply it to the propagation of acceleration waves in elastoplastic materials undergoing finite deformation.

Our finite-strain theory of plasticity is based on a corresponding study in the context of small strains [26], and has the feature of embracing the classical notions of convex yield surfaces and the normality law; the evolution law is quite general, though, and is developed within the framework of non-smooth convex analysis. As mentioned above, we apply our constitutive theory to a study of acceleration waves in elastic-plastic solids undergoing finite deformation.

The rest of this Chapter is structured as follows:

- (i) we review the relevant contributions to the finite-strain theory of plasticity;
- (ii) we summarize the contributions to the study of acceleration waves in elastic-plastic materials;
- (iii) we present the objectives of this thesis; and,
- (iv) we summarize the notation that will be in use.

Finite Strain Theories of Plasticity

Green and Naghdi formulated a theory in 1965 [27], which was soundly based on the principles of thermodynamics. No assumptions on the smallness of strains or symmetry properties of materials were made. The fundamental kinematic quantities were total strain and plastic strain. Hence, an additive decomposition of the Lagrangian strain tensor into its elastic and plastic parts was proposed. The plastic strain tensor was assumed to depend entirely on the history of deformation in a prescribed manner. The general yield function was assumed to be a function of stress, plastic strain and temperature, and it was assumed to be continuously differentiable. The evolution law was expressed in terms of the time rate of plastic strain. The corresponding theory for infinitesimal strains was obtained as a special case.

In contrast to the additive decomposition of strain into its elastic and plastic parts, Lee [40] in 1969 proposed a multiplicative decomposition of the deformation gradient tensor into its elastic and plastic parts. The multiplicative decomposition has proved more popular, and forms the basis for a number of alternative theories; as examples we mention the works of Mandel [48], Halphen and Nguyen [29], Simo and Ortiz [75], Simo [74], and of Moran, Ortiz and Shih [55]. Kim and Oden [35,36,37] have, in a series of papers, discussed other forms of decomposition of the deformation. An important alternative, in use for some time now, is the decomposition of deformation rate and its elastic and plastic parts [32,59].

In Lee's paper [40], the plastic part of the deformation gradient is assumed to be a functional which takes into account the past history of plastic flow, while the elastic part is a function of the stress. Lee incorporates in his theory the classical theory of isotropic work hardening, generalized to include additional features which a finite-strain theory requires. In [41], Lee investigates the relationship between the multiplicative decomposition, and the additive decomposition of deformation rate. In [28] Green and Naghdi discuss further the issue of elastic-plastic deformation at finite-strain, and recover Lee's results for isotropy as a special case.

A key aspect of any theory of plasticity, and in particular of a finite-strain theory, is the form taken by the evolution equation or flow law. One approach which for the finite-strain case can be traced back to the works of Mandel [48], Halphen and Nguyen [29] and Rice [67], has been to develop a theory within the framework of an internal variable theory, and to express the evolution equation in terms of a potential. In [48] Mandel applies the theory of continuum thermodynamics with internal variables to materials which undergo plastic deformation. He expresses the plastic strain tensor in terms of a plastic potential function and certain scalar parameters. Halphen and Nguyen [29] express the evolution law as a normality law and dually in terms of a dissipation function. In [67] Rice shows that the kinetics of the structural rearrangements of the material may be characterized by a scalar potential function of the macroscopic stress-state at each instant in the history of deformation, for a time-dependent material. In the case of time-independent

materials he recovers the classical normality rule with respect to a yield surface in stress space.

In [73] Simo uses an incremental formulation which is based on an additive decomposition of the deformation rate with the multiplicative decomposition of deformation gradient. In [75] Simo and Ortiz consider a unified approach to finite deformation elastoplasticity which embodies both additive and multiplicative decomposition theories, although they take the multiplicative decomposition of the deformation gradient as the basis of their theory. Simo [74] proposes a theory of finite strain plasticity based on the maximum plastic work inequality. Here the dissipation is maximised amongst all deformations, holding the internal variable constant. The constitutive equations are finally reduced to a form in which the free energy depends on the total and plastic (right) Cauchy-Green strain tensors. After postulating the yield condition, he obtains the corresponding flow rule through an extension of the classical principle of maximum plastic dissipation.

Kim and Oden [35,36,37] use generalized stress potentials for deriving constitutive equations for the rate of plastic deformation and the rate of change of an internal variable. This theory takes as its basis the methods of non-smooth, non-convex analysis and does not require the existence of yield functions, though they can enter the theory through the definition of flow potentials, which do not have to be convex or differentiable. A general potential functional

determines the plastic deformation rate and also the evolution of the internal state variables.

Dashner [20,21,22] formulates a theory of isotropic, rate-independent, strain-hardening elastoplasticity based on the invariance requirements of elastic properties, as an incremental state variable theory. In [20] invariance of elastic properties is assumed under continued plastic flow. This leads to the restriction on the class of admissible energy functions and allows for a restatement of the known stability requirements. In this theory, a convex yield function is used which depends only on the invariants of Kirchhoff stress. This function acts as a potential for the plastic strain increments. Symmetric positive definite deformation tensors are used rather than deformation gradients. The right Cauchy-Green strain tensor is decomposed into its elastic and plastic parts in an additive form, and incompressibility of plastic flow is imposed by setting the determinant of the plastic part equal to unity. In [21] an Eulerian formulation is considered. The state of plastic deformation is described in terms of a scalar work hardening parameter and a single finite-strain elastic deformation tensor that fixes the elastic stretch ellipsoid in the current configuration. In [22] the rotational invariance requirement is imposed in the intermediate unstressed configuration.

Lubliner [45] shows that in finite deformation plasticity, the maximum dissipation postulate is equivalent to a six dimensional normality rule as in small deformation plasticity. The material

plastic strain rate is normal to the yield surface in the space of the second Piola-Kirchhoff stress. He states that any postulated flow rule for the plastic spin is an additional hypothesis which is independent of the maximum dissipation principle.

Naghdi and Trapp [56] derive a special class of response functions for the constitutive equations in the nonlinear isothermal theory of elastic-plastic materials. This theory, for isotropic materials, is based on a thermodynamical theory. Constitutive equations for field quantities such as the free energy and stress are given. It is found that the free energy response is a quadratic function of the elastic strain tensor. In [57] Naghdi and Trapp confine their attention to the purely mechanical theory of finitely deformed elastic-plastic materials with the aim of obtaining rate type nonlinear constitutive equations. Their theory is based on the physically plausible assumption of non-negativity of work in a closed cycle of deformation. They express the constitutive equation for the plastic strain rate in terms of the yield function and the functions representing the stress response and work hardening. In [58] Naghdi and Trapp prove that, in the presence of finite deformation, the restrictions which are obtained from certain inequalities found in the two previous papers imply the normality of the plastic strain rate and the convexity of yield surfaces in stress space for a special class of elastic-plastic and rigid-plastic materials.

Kleiber [38] defines properly invariant forms of strain and rate of strain measures in the reference, intermediate and current

configurations. He demonstrates that the commonly accepted multiplicative decomposition of deformation gradient leads in a natural way to the concept of the additive decomposition of strain measures. He further points out that this additive decomposition is valid only if suitable definitions of stress are used.

Lubarda and Lee [44] indicate in their theory that the plastic part of an elastic-plastic deformation is that which remains when the stress and naturally the elastic strain, are reduced to zero. They use the multiplicative decomposition of deformation gradient and clarify the notion that the plastic part of the deformation gradient can be associated with an intermediate configuration only in a local sense.

Carroll [6] proposes a rate-independent constitutive theory for finite inelastic theory in terms of the symmetric Piola-Kirchhoff stress, the Lagrangian strain and a kinematic tensor which takes inelastic or microstructural effects into account. His assumptions of continuity in the transition from loading to neutral loading, consistency, and the non-negativity of work in a closed cycle of deformation lead to simplifications of the theory. The response is described by two scalar functions: a stress potential and a yield function. This theory can describe an isotropic or anisotropic response and allows hardening, softening or ideal behaviour. The assumption of the non-negativity of work in a closed strain cycle reduces the constitutive description to one in terms of a stress potential and a yield function.

Nemat-Nasser [60] examines the problem of relating the decomposition of total strain, into its elastic and plastic parts, to the multiplicative decomposition of deformation gradient, and to the additive decomposition of deformation rate. After certain clarifications on the nature of elastic and plastic parts of deformation gradient he points out that while one cannot additively decompose Lagrangian strain, the corresponding strain rate can be decomposed in that manner. In [61] Nemat-Nasser indicates that all existing elastoplasticity concepts lead to total strain measures which can be additively decomposed into their elastic and plastic parts, provided that the corresponding elastic and plastic strain rates are conjugate to the same stress measure. Nemat-Nasser shows that Lee's theory also leads to an additive decomposition of the deformation rate (Eulerian strain rate). In [62] he examines isotropic and kinematic hardening in elastic-plastic materials.

Casey and Naghdi [8] use a multiplicative decomposition of the deformation gradient and point out that elastic and plastic constituents of the deformation gradient are not unique. In [9] they propose a purely mechanical rate-type theory of elastic-plastic materials using a strain space formulation, and characterize the strain-hardening behaviour by rate-independent quantities occurring in the yield criterion in strain space. In [10] the authors stress the importance of objectivity and state that configurations which differ from each other by only rigid body displacements are equivalent and are physically indistinguishable. In [11] Casey and Naghdi show that their scalar function which characterizes strain-

hardening behaviour is a scalar invariant of a certain fourth-order tensor which plays a fundamental role in the theory of plasticity. They also discuss the invertibility of the relationship between stress rate and strain rate. In [12], which is an extension of the abovementioned contribution, the authors examine the normality and convexity conditions. In [7] Casey and Naghdi use a strain-space formulation for rigid-plastic materials and consider a scalar hardening parameter. They find that in their case Lagrangian and Eulerian descriptions are equivalent, and the particular choice of an objective stress rate is immaterial [14].

Sewell [82,83] examines the plastic flow at vertices of a yield surface and finds that "a yield-surface corner lowers the buckling stress of an elastic-plastic plate under compression". We will present aspects of Sewell's theory in detail in Section 3.6 before incorporating it into our theory.

Another work of interest and relevance to our own, is the contribution of Antman and Szymczak [1]. These authors use an internal variable approach to develop a constitutive theory for elastic-plastic materials undergoing finite deformations. They then make use of this theory in a numerical investigation of shock waves in problems characterized by one spatial dimension, that is, longitudinal motions of elastic bars. Here, the constitutive equations are assumed to be rate-independent and the yield surface is expressed in strain space.

Eve, Reddy and Rockafellar [26] explore the nature of the evolution equation in internal variable formulations of small-strain elastoplasticity in greater depth by using the methods of convex analysis. The authors consider the evolution equation in a form in which the thermodynamic force belongs to a set defined by a certain multi-valued map. They show that an assumption called maximal responsiveness, which is closely related to the classical maximum plastic inequality, is necessary and sufficient for the existence of a dissipation function and yield surface with the requisite properties. They also investigate the relationship between the yield *function* and dissipation function. The theory is illustrated with various examples.

Eve, Gültop and Reddy extend [26] in [25] to cover the case of elastoplastic bodies undergoing arbitrarily large deformations. This theory is based on the classical assumptions of a convex region of admissible stresses and the normality law. The overall effects of plastic behaviour are contained in the theory by the use of one or more internal variables. The problem of the decomposition of total deformation into its elastic and plastic parts is resolved by using a multiplicative split of deformation gradient. The thermodynamic restrictions arising from the second law, and the use of the methods of convex analysis, lead in a natural way to the evolution equation. A concrete example of a free energy is discussed in detail to illustrate the theory. We present this theory in Chapter III in detail together with extensions which do not appear in [25].

Acceleration Waves in Elastic-Plastic Solids

The earliest contributions to the formulation of the propagation of acceleration waves in elastic-plastic solids were made by Hill [30] and Mandel [47], both in 1962. In [30], Hill used a purely mechanical theory of elastoplasticity to investigate the propagation of acceleration waves. He expressed the Kirchhoff stress rate in terms of strain rate by a homogeneous linear tensor relation between them, and used the normality rule of classical plasticity. The author used the theory of singular surfaces and obtained the propagation condition of longitudinal and transverse waves. He obtained a matrix equation for the possible wave speeds, hence the wave propagation problem was reduced to the form of a characteristic value problem. In [47], Mandel used a similar approach and obtained the propagation condition of acceleration waves in elastic-plastic solids undergoing small deformations. Later Mandel extended his results in [49] to elastic-plastic solids undergoing finite deformations. In both [47] and [49] Mandel derived a set of inequalities which allow wave speeds in elastic and plastic materials to be ordered. In [49], Mandel developed a thermodynamic theory of waves in elastic-plastic conductors and non-conductors.

Balaban, Green and Naghdi [3] investigated the propagation of acceleration waves in elastic-plastic solids undergoing finite deformations on the basis of a nonlinear thermodynamical theory of elastic-plastic continua. By the use of the theory of singular surfaces, the authors obtained the propagation condition of

acceleration waves in the form of a characteristic value problem for the case of plastic loading on both sides of the wave front. Later on, they investigated conditions for the existence of real wave speeds by using the properties of the matrices. Following the technique used by Mandel, the authors were able to compare the elastic and plastic wave speeds. They also examined the conditions for the propagation of the fronts of loading and unloading, that is, waves which are in the form of elastic-plastic interfaces. The authors also investigated the propagation of higher order waves in plastic media, and concluded their contribution with a concrete example of plastic waves in uniaxial motion.

Raniecki [66] formulated the propagation of acceleration waves in rate-independent elastic-plastic conductors and non-conductors undergoing small deformations. He obtained the propagation conditions for plastic waves and elastic-plastic interfaces, and compared the elastic and plastic wave speeds by the use of Mandel's inequalities which he derived from his symmetric acoustic tensor. He also examined simple waves and higher order waves. Unlike [3], Raniecki obtained a propagation condition in the form of a characteristic value problem by the use of certain assumptions, which we discuss in Section 5.3.

In a series of papers [63,64,65], Piau investigates the propagation of acceleration waves in elastic-plastic materials. She obtains the propagation condition and the speeds of propagation in [63], and extends her investigation in [64] to cover the propagation

of acceleration waves in elastic-plastic solids undergoing finite deformations. The author compares the elastic and plastic wave speeds in [64] by using the general method of Mandel. d'Eschata [23] examines the properties of the plastic acoustic tensor and the propagation conditions for loading and unloading waves.

Kosinski and Peryzna's analysis [39] of acceleration waves in materials with internal parameters is not explicitly related to wave propagation in plastic materials, but displays certain similarities with the treatment of waves in plastic materials as their constitutive theory is formulated with the use of internal parameters. Each internal parameter yields an unknown term in the fundamental equation of motion of acceleration waves. The information used for the solution of the additional unknowns is provided by the constitutive theory. The authors examine conditions for the propagation of homothermal, isothermal and thermal waves.

Janssen, Datta and Jahsman [33] investigate the propagation of acceleration waves in an elastic-plastic strain-hardening rate-independent solid undergoing small strains. They obtain expressions for wave speeds and the transport (growth and decay) equations, with specific applications to plane and cylindrical waves.

Ting [76] investigates the propagation of shock waves and acceleration waves in non-homogeneous and anisotropic solid media undergoing small strains. He assumes that the elastic response is linear whereas the plastic response is nonlinear. He obtains the

acoustic tensor, plastic wave speeds and the speeds of elastic-plastic interfaces and compares the elastic and plastic wave speeds. The author [77] investigates the plastic wave speeds in elastic-plastic materials with isotropic hardening and obtains plastic wave speeds in materials with von-Mises and Tresca yield criteria. However, the wave speeds in a Tresca material are obtained without taking account of the influence of non-smoothness of the yield surface.

Lee and Wierzbicki [43] study the propagation of shock waves in elastic-plastic solids undergoing finite strains. In [42] a similar study is made by Lee and Liu for plane waves. A multiplicative decomposition of the deformation gradient is made in both contributions. Among other contributions to the field of wave propagation in elastic-plastic solids (not necessarily acceleration waves) we can mention the papers by Shack [72] who examines infinitesimal plane waves propagating through a deformed material, by Kenning [34] who investigates the existence of spherically-symmetric elastic-plastic interface motions, and by Milne, Morland and Yeung [54] who examine spherically-symmetric plastic waves in solids undergoing infinitesimal strains.

Objectives of the Thesis

We first aim to develop a theory of finite strain plasticity within the framework of convex analysis, and then to apply this theory to the propagation of acceleration waves. In Chapter II we

present a review of results from convex analysis and make a connection between certain results of convex analysis, and some basic notions of the theory of plasticity.

One of the basic problems in the study of the kinematics of elastic-plastic continua is the manner in which total deformation is decomposed into its elastic and plastic parts. In this thesis we will use Lee's multiplicative decomposition [40] of the deformation gradient and discuss the results of the use of this method of decomposition. We will need the spectral representations of Cauchy-Green tensors for later use, hence we summarise the spectral forms of their elastic and plastic parts of deformation gradient. Other kinematic quantities to be discussed will be the velocity gradient, deformation rate and spin tensors.

In the development of the theory in Chapter III we will confine attention to isothermal behaviour; therefore we will reduce the equation of the balance of energy and Clausius-Duhem inequality to those forms which are appropriate for such behaviour. We will introduce a set of internal variables $\{\gamma, \mathbf{A}\}$ which describe overall effects associated with plastic deformation; γ is a scalar, and \mathbf{A} a second rank symmetric tensor. We will consider a set of constitutive equations in which free energy and stress will both be functions of the elastic part of the deformation gradient and the set of internal variables. The evolution equation, which describes the evolution of inelastic behaviour, will also be assumed to be a function of the variables of free energy. The evolution equation, more precisely,

will describe the time rates of the plastic part of deformation gradient and the set of internal variables. We will apply the principle of material frame indifference to the set of constitutive equations for the free energy and stress, and obtain reduced forms for their representations. We will impose the restriction that there is no volume change accompanied by plastic deformations, so that volume changes arise solely due to the elastic parts of deformation. We will obtain the explicit form of the evolution equation and its relation to the yield function. Here we will consider a convex, but non-smooth yield surface; therefore we will obtain the normality rule in a general form. Then we will consider an example of a free energy function of an elastic-plastic material which exhibits both kinematic and isotropic hardening, and obtain the explicit form of the yield function and flow law.

For the study of acceleration waves we will later on need a specific equation for the normality condition, at yield surface vertices; for this purpose it will prove useful to incorporate Sewell's theory [70,71] into our theory.

The second part of the thesis will basically be devoted to the study of acceleration waves, in the context of the constitutive theory developed here. Therefore we will review the basic ideas of the theory of singular surfaces and acceleration waves in Chapter IV.

In Chapter V we will obtain the jump form of the equation of the balance of linear momentum in terms of the time rate of stress

across a second order singular surface, that is, an acceleration wave front. The time rate of stress whose jump across the wave front will be required, will be obtained from the results of our constitutive theory which takes yield surface vertices into account. Jumps of further variables such as the velocity gradient, spin and deformation rate tensors as well as the scalar parameters in the expression of the normality rule will be obtained and substituted in the jump form of the equation of the balance of linear momentum. The aim will be to reduce the problem of acceleration wave propagation to that of a characteristic value problem, hence to obtain a positive-definite and symmetric acoustic tensor. We will consider plastic waves, that is, acceleration waves which propagate in a region where the solid body undergoes plastic deformation both ahead and behind the wave front. After obtaining the acoustic tensor for elastic-plastic materials with non-smooth yield surfaces, we will obtain the acoustic tensor for materials with smooth yield surfaces as a special case. We will then obtain the propagation conditions for elastic-plastic interfaces, as well as a general basis for the comparison of elastic and plastic wave speeds by treating the acoustic tensor in a way similar to that of Mandel [47,49].

We will obtain the propagation conditions for longitudinal and transverse principal waves. The assumption of isotropy will enable us to express the yield functions in principal stress space, and the free energy in terms of elastic principal stretches. We will compare the speeds of propagation in isotropic elastic-plastic materials with the von Mises and Tresca yield criteria, using a specific free energy

function for compressible neo-Hookean materials, for the case of uniaxial stress.

Notation

The space of tensors of rank n is denoted by T^n ; the special cases of vectors ($n = 1$) and scalars ($n = 0$) are respectively denoted by V and R . We generally use bold faced upper case characters for tensors, bold faced lower case characters for vectors, and ordinary lower case characters for scalars.

A tensor T of rank n is an n -linear function on $T^n \equiv V \times \cdots \times V$ (n times). A member of T^n may be constructed by defining the n -fold tensor product $u_1 \otimes u_2 \otimes \cdots \otimes u_n$, for $u_i \in V$, by

$$u_1 \otimes u_2 \otimes \cdots \otimes u_n (a_1, a_2, \dots, a_n) = \prod_{i=1}^n (u_i \cdot a_i)$$

for all $a_i \in V$.

If e_i is an orthonormal basis in V then

$$\underbrace{e_i \otimes e_j \otimes \cdots}_{n\text{-terms}}$$

is a basis for T^n . Then for the case $n = 2$, for example, a member T of T^2 has the representation

$$T = T_{ij} e_i \otimes e_j ,$$

where the components T_{ij} of \mathbf{T} are found from

$$T_{ij} = \mathbf{T}(\mathbf{e}_i, \mathbf{e}_j) \quad .$$

We can also regard tensors as linear vector- or tensor-valued maps. For example, the second-rank tensor $\mathbf{A} = \mathbf{u} \otimes \mathbf{v}$ may be defined by

$$\mathbf{A} : V \rightarrow V \quad , \quad (\mathbf{u} \otimes \mathbf{v})(\mathbf{a}) = \mathbf{u}(\mathbf{v} \cdot \mathbf{a}) \quad ,$$

and the fourth-rank tensor $\mathbf{H} = \mathbf{u} \otimes \mathbf{v} \otimes \mathbf{w} \otimes \mathbf{x}$ by

$$\mathbf{H} : T^2 \rightarrow T^2 \quad , \quad \mathbf{u} \otimes \mathbf{v} \otimes \mathbf{w} \otimes \mathbf{x} (\mathbf{a} \otimes \mathbf{b}) = (\mathbf{w} \cdot \mathbf{a})(\mathbf{x} \cdot \mathbf{b}) \mathbf{u} \otimes \mathbf{v} \quad .$$

Using these definitions, general representations for expressions such as $\mathbf{A}\mathbf{a} \in V$ and $\mathbf{H}\mathbf{A} \in T^2$ (for $\mathbf{H} \in T^4$ and $\mathbf{A} \in T^2$) are readily derived.

We use the summation convention for the indices of vectors and tensors in the following form:

- (i) if an index appears on both sides of an equation it is not summed;
- (ii) indices which occur twice in an expression on only one side of an equation are summed;
- (iii) the summation is explicitly shown in cases where indices occur more than twice in an expression.

For example, summation is not implied over i or j in $B_{ij} = K_{ijkl} C_{kl}$, but summation is implied over both k and ℓ . Summation is implied in $F = F_{kL} e_k \otimes e_L$ over both k and L , but it is explicitly shown in the expression

$$C = \sum_{i=1}^3 c_i p_i \otimes p_i \quad .$$

CHAPTER II

REVIEW OF RESULTS FROM CONVEX ANALYSIS

The constitutive theory for finite strain behaviour of elastic-plastic materials which will be presented in this thesis is based on the classical assumption of a convex region of admissible stresses. We will exploit this assumption by embedding the theory within the framework of convex analysis; this framework will serve the purpose of facilitating a unified treatment of the theory, and of clarifying the relationships between various concepts in the elastic-plastic constitutive theory.

In this Chapter we review those results from convex analysis which will be of use later. The standard reference is the text by Rockafellar [69], though the papers [26] and [25] also contain, in summarised form, most of the results relevant to plasticity theory.

We denote by \mathbb{R} the set of real numbers and by \mathbb{R}^n the linear space of n -tuples of real numbers. Throughout this Chapter E will denote a finite dimensional space, isomorphic to \mathbb{R}^n for appropriate n , and K a subset of E .

We adopt the commonly used notation to denote the following types of intervals on the real line \mathbb{R} . If we take two points a and b

in \mathbb{R} such that $a \leq b$, then we define

- (i) the open interval $(a,b) = \{x : x \in \mathbb{R}, a < x < b\}$;
- (ii) the closed interval $[a,b] = \{x : x \in \mathbb{R}, a \leq x \leq b\}$;
- (iii) the half-open intervals $(a,b] = \{x : x \in \mathbb{R}, a < x \leq b\}$
and $[a,b) = \{x : x \in \mathbb{R}, a \leq x < b\}$.

Functions considered in this Chapter as well as those in the subsequent Chapters might attain the values $\pm \infty$. Hence we state here the adopted rules for calculations involving $+\infty$ and $-\infty$, which are the following:

- (i) $a + \infty = \infty + a = \infty$ for $-\infty < a \leq +\infty$;
- (ii) $a - \infty = -\infty + a = -\infty$ for $-\infty \leq a < \infty$;
- (iii) $a\infty = \infty a = \infty$, $a(-\infty) = (-\infty)a = -\infty$ for $0 < a \leq \infty$;
- (iv) $a\infty = \infty a = -\infty$, $a(-\infty) = (-\infty)a = \infty$ for $-\infty \leq a < 0$;
- (v) $0\infty = \infty 0 = 0(-\infty) = (-\infty)(0) = 0$, $-(-\infty) = \infty$;
- (vi) $\inf \emptyset = +\infty$, $\sup \emptyset = -\infty$.

For any function $f : E \rightarrow \bar{\mathbb{R}}$ with values in $\bar{\mathbb{R}} = \mathbb{R} \cup \{\pm\infty\}$, the *effective domain*, $\text{dom } f$ is the set of points x in E for which $f(x)$ is finite:

$$\text{dom } f = \{x \in E : f(x) < \infty\} .$$

Convex Functions

The function f is *convex* if

$$f(\theta x + (1 - \theta)y) \leq \theta f(x) + (1 - \theta)f(y) \quad (2.1)$$

for all $x, y \in E$ and $0 < \theta < 1$. Furthermore f is *strictly convex* if in (2.1) we have the strict inequality:

$$f(\theta x + (1 - \theta)y) < \theta f(x) + (1 - \theta)f(y) \quad (2.2)$$

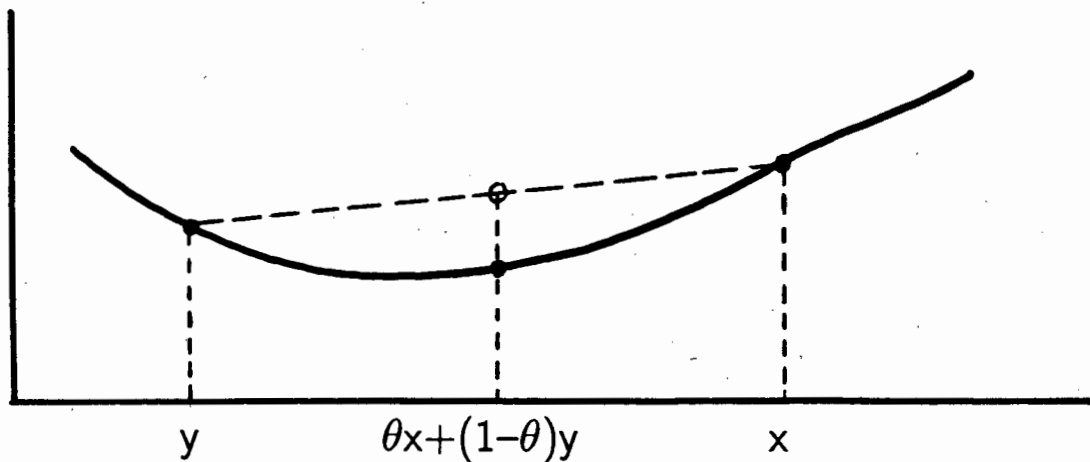
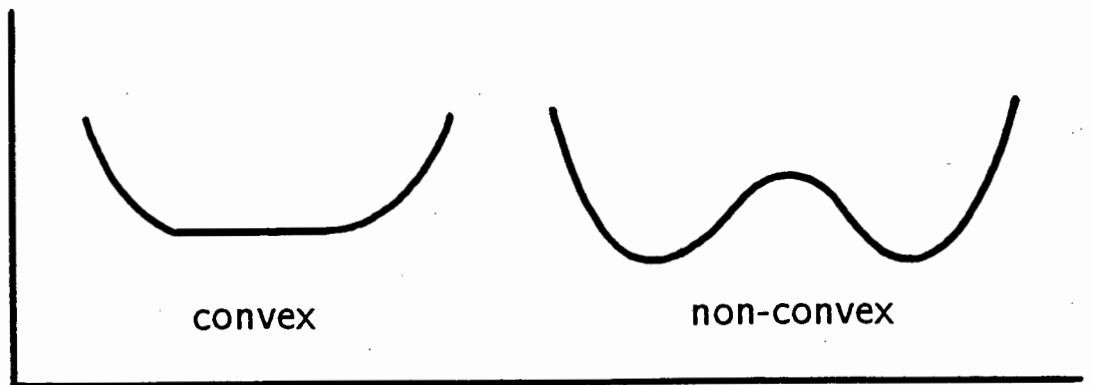


Figure 2.1 Convex, non-convex and strictly convex functions in \mathbb{R} .

A function f is *lower semicontinuous* (see Figure 2.2) if

$$\liminf_{n \rightarrow \infty} f(x_n) \geq f(x_0) \quad ,$$

for any sequence $\{x_n\}$ converging to x_0 , and *positively homogeneous* (see Figure 2.3) if

$$f(ax) = a f(x) \quad \text{for all } x \in E \quad , \quad 0 < a \in \mathbb{R} \quad .$$

The *epigraph* of f , denoted $\text{epi } f$, is the set of ordered pairs

$$\text{epi } f = \{(x, a) \in E \times \mathbb{R} : f(x) \leq a\} \quad ,$$

and is illustrated in Figure 2.3.

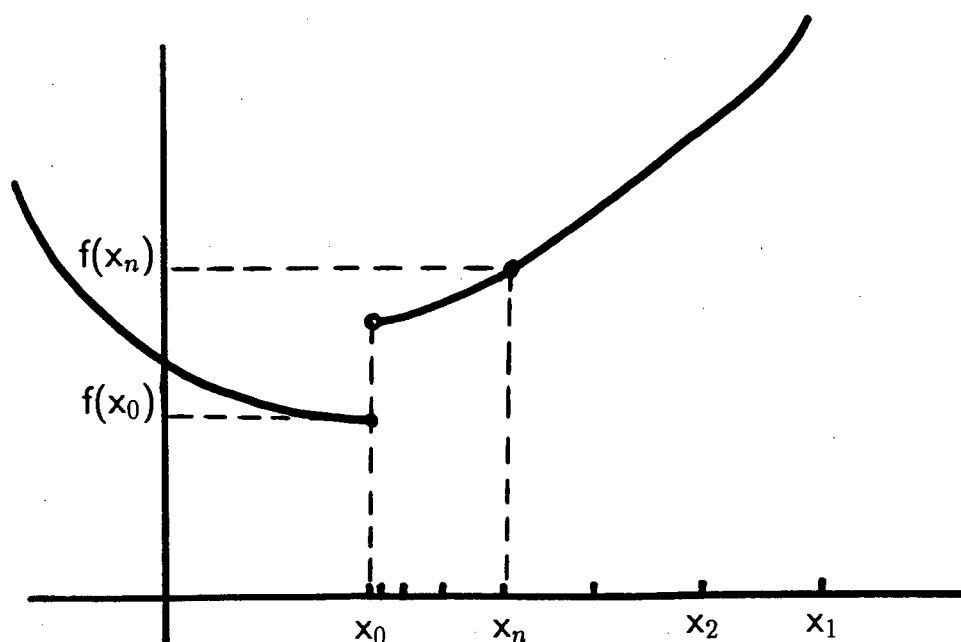


Figure 2.2 A lower semicontinuous function f for the sequence $\{x_n\}$ converging to x_0 .

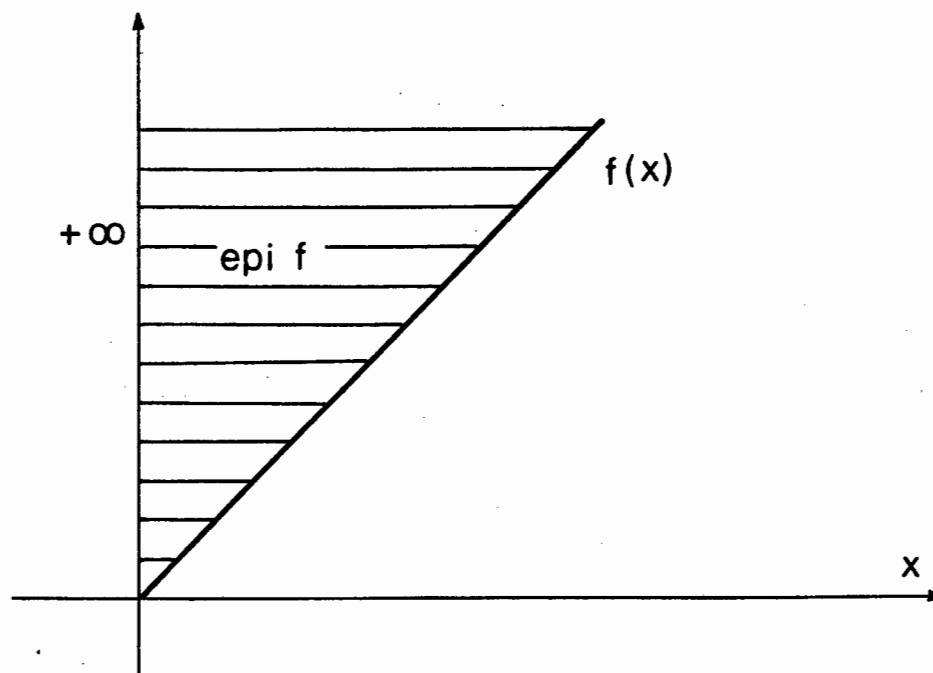


Figure 2.3 The epigraph $\text{epi } f$ of the positively homogeneous function $f(x) = x, x \geq 0, f(x) = +\infty, x < 0$, $\text{dom } f = [0, \infty), \text{epi } f = \{(x, a) : x \geq 0; x \leq a\}$.

A convex function is said to be *proper* if $\text{epi } f$ is non-empty and contains no vertical lines, that is, if $f(x) < +\infty$ for at least one x and $f(x) > -\infty$ for every x .

A function $f : E \times \mathbb{R}$ is called a *gauge* if

- (i) $f(x) \geq 0$ for all $x \in E$;
- (ii) $f(0) = 0$;
- (iii) f is convex, positively homogeneous and lower semi-continuous.

A multivalued or set-valued map $H : p \rightarrow H(p) \in E$ is said to be *responsive* if

- (i) $0 \in H(0)$;
- (ii) for any $p_0, p_1 \in E$

$$(x_0 - x_1) \cdot p_0 \geq 0 \quad \text{and} \quad (x_1 - x_0) \cdot p_1 \geq 0 \quad (2.5)$$

for all $x_0 \in H(p_0)$, $x_1 \in H(p_1)$.

A responsive map H is said to be *maximal responsive* if there is no other responsive map whose graph properly includes the graph of H . This definition here is equivalent to the following:

The map H is *maximal responsive* if it is responsive, and if, given p_0 and $x_0 \in H(p_0)$, the inequalities (2.5) imply that $x_1 \in H(p_1)$.

We state the following theorem, omitting its proof which is given in [26].

THEOREM 2.1 The set-valued map H is maximal responsive *if and only if* there exists a gauge D which has the following property:

$$H(p) = \partial D(p) \quad \text{for all } p \in E \quad .$$

When \mathbb{H} is maximal responsive it determines D uniquely and

$$D(\mathbf{p}) = \begin{cases} \mathbf{x} \cdot \mathbf{p} & \text{for all } \mathbf{x} \in \mathbb{H}(\mathbf{p}) \quad , \quad \text{if } \mathbb{H}(\mathbf{p}) \neq \emptyset \quad , \\ +\infty & \text{otherwise.} \end{cases} \quad (2.6)$$

The notion of maximal responsiveness is used to make a connection with the classical maximum plastic work inequality [26]. The gauge function (2.6) is, in that context, the *dissipation function* $D(\mathbf{p})$ which gives the rate of plastic work. Because of the dissipative nature of the plastic work, the assumption about its non-negativity is a physical requirement, particularly dictated by the second law of thermodynamics. (See Chapter III on the constitutive theory of elastic-plastic materials.)

Convex Sets

We define the *neighbourhood* of a point \mathbf{c} in E to be the set

$$N(\mathbf{c}, \epsilon) = \{\mathbf{x} : \mathbf{x} \in E \quad , \quad |\mathbf{x} - \mathbf{c}| < \epsilon\} \quad (2.7)$$

where ϵ is the radius of the neighbourhood and the distance $|\mathbf{x} - \mathbf{c}|$ from \mathbf{x} to \mathbf{c} is given by

$$|\mathbf{x} - \mathbf{c}| = \sqrt{(x_1 - c_1)^2 + (x_2 - c_2)^2 + \dots + (x_n - c_n)^2} \quad .$$

The point c in E is called an *interior point* of a set X of points in E if we can always find a neighbourhood of c , all of whose points belong to X . The set of all interior points of a set is denoted by $\text{int } X$. A subset X of E is an *open set* if every point of X is an interior point. The *boundary* of a subset X of E , denoted by $\text{bd } X$ is the set of all points in X excluding the interior points of X .

If X is a subset of E and c is a point in E (not necessarily in X), then c is called a *limit point* of X if every neighbour of c contains at least one point in X distinct from c . Then, if the set X contains all of its limit points it is called a *closed set*. We define the *closure* \bar{X} of a set X in E to be the union of X and all its limit points. Thus, $\bar{X} = X \cup \{\text{limit points}\}$ is a closed set.

Let K be a subset of E . K is a *convex set* if

$$\theta x + (1 - \theta)y \in K$$

for any $x, y \in K$ and $0 < \theta < 1$. A convex set and a non-convex set are illustrated in Figure 2.5.

The *normal cone* $N_K(x)$ to a convex set K at x is the set (see Figure 2.6)

$$N_K(x) = \{p \in E : (y - x) \cdot p \leq 0 \text{ for all } y \in K\} \quad (2.8)$$

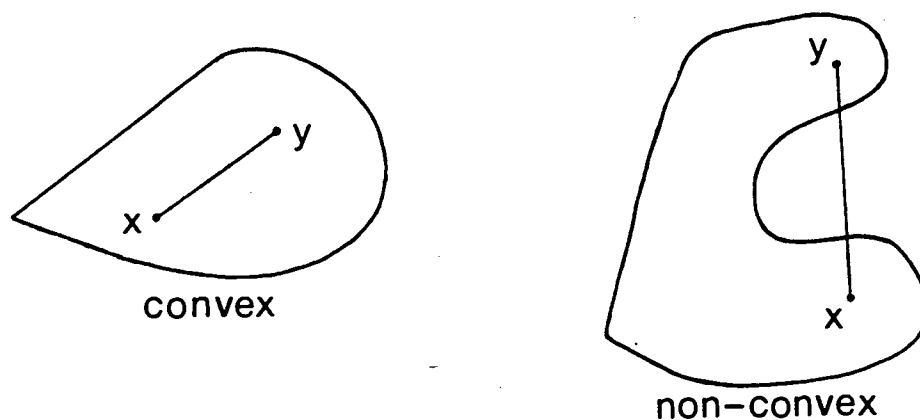


Figure 2.5 Examples of convex and non-convex sets in \mathbb{R}^2 .

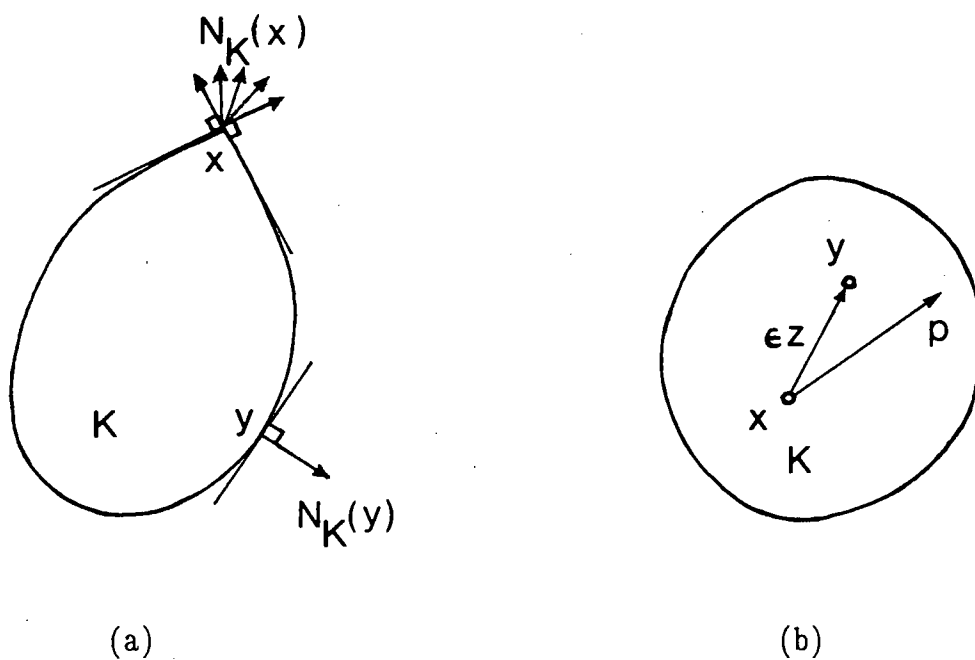


Figure 2.6 (a) The normal cone $N_K(x)$ to a convex set K at a non-smooth point x and at a smooth point y .
 (b) The consideration of the normal cone element p at an interior point x which yields $p = 0$.

If we consider any point \mathbf{x} in the interior of K and take $\mathbf{y} = \mathbf{x} + \epsilon \mathbf{z}$ (2.8) becomes

$$N_K(\mathbf{x}) = \{\mathbf{p} \in E : \epsilon \mathbf{z} \cdot \mathbf{p} \leq 0 \text{ for all } \mathbf{y} \in K\}$$

$$\Rightarrow \mathbf{p} = \mathbf{0} \quad \text{and} \quad N_K(\mathbf{x}) = \{\mathbf{0}\} \quad .$$

For a point \mathbf{x} on the boundary of K , $N_K(\mathbf{x})$ is the cone of outward normals at that point. We define a cone with vertex zero to be a subset K of E which is closed under positive scalar multiplication, that is $a\mathbf{x} \in K$ when $\mathbf{x} \in K$ and $a \geq 0$:

$$K = \{\mathbf{x} : \mathbf{x} \in K \Rightarrow a\mathbf{x} \in K \text{ for all } a \geq 0\} \quad .$$

The set K is a union of half-lines emanating from the origin.

The *indicator function* I_K of any set K which is a subset of E is defined by

$$I_K(\mathbf{x}) = \begin{cases} 0 & \text{if } \mathbf{x} \in K \\ +\infty & \text{otherwise.} \end{cases} \quad (2.9)$$

The indicator function provides one of the useful correspondences between convex sets and convex functions. The epigraph of the indicator function is a half cylinder with cross-section K . K is a convex set if and only if $I_K(\mathbf{x})$ is a convex function on E . The indicator function is lower semicontinuous.

Conjugate Functions

For any convex function $f : E \rightarrow \bar{\mathbb{R}}$, the Legendre-Fenchel *conjugate function* f^* of f is defined by

$$\begin{aligned} f^* : E &\rightarrow \bar{\mathbb{R}} \quad , \\ f^*(x^*) &= \sup\{x^* \cdot x - f(x) : x \in E\} \quad . \end{aligned} \quad (2.10)$$

For any convex and lower semicontinuous proper function f we have the property

$$f = (f^*)^* \quad .$$

The *support function* $D_K : E \rightarrow \bar{\mathbb{R}}$ of a convex set K is defined by

$$D_K(x^*) = \sup\{x^* \cdot x : x \in K\} \quad . \quad (2.11)$$

The support function is convex, lower semicontinuous and positively homogeneous. From the definitions (2.10) and (2.11) we see that the support function D_K is conjugate to the indicator function I_K :

$$I_K^* = D_K \quad . \quad (2.12)$$

For particular interest here we should note that, I_K is convex and lower semicontinuous when K is a closed and convex set. It

follows that:

$$I_K = D_K^* = \left[I_K^* \right]^* . \quad (2.13)$$

We have the following important result in relation to the definition (2.3) of the *subdifferential*:

$$x^* \in \partial f(x) \quad \text{if and only if} \quad x \in \partial f^*(x^*) .$$

Given the closed set K , we define a function $f : E \rightarrow \bar{\mathbb{R}}$ by

$$f(x) = \inf\{\mu \geq 0 : x \in \mu K\} \quad (2.14)$$

where $\mu K = \{\mu x : x \in K\}$. From (2.14) it is seen that f is a *non-negative , positively homogeneous convex* function whose level set at 1 is equal to K (see Figure 2.7); in other words f is a *gauge*.

The *polar* f^0 of a gauge function is defined by

$$f^0(x^*) = \inf\{\mu \geq 0 : x \cdot x^* \leq \mu f(x) \quad \text{for all } x\} . \quad (2.15)$$

If f is finite everywhere and positive except at the origin, (2.15) becomes

$$f^0(x^*) = \sup_{x \neq 0} \frac{x \cdot x^*}{f(x)} . \quad (2.16)$$

If f is a gauge function, then the polar f° of f is a proper lower semi-continuous gauge function, and $(f^\circ)^\circ = f$.

The following well-known result is taken from [69].

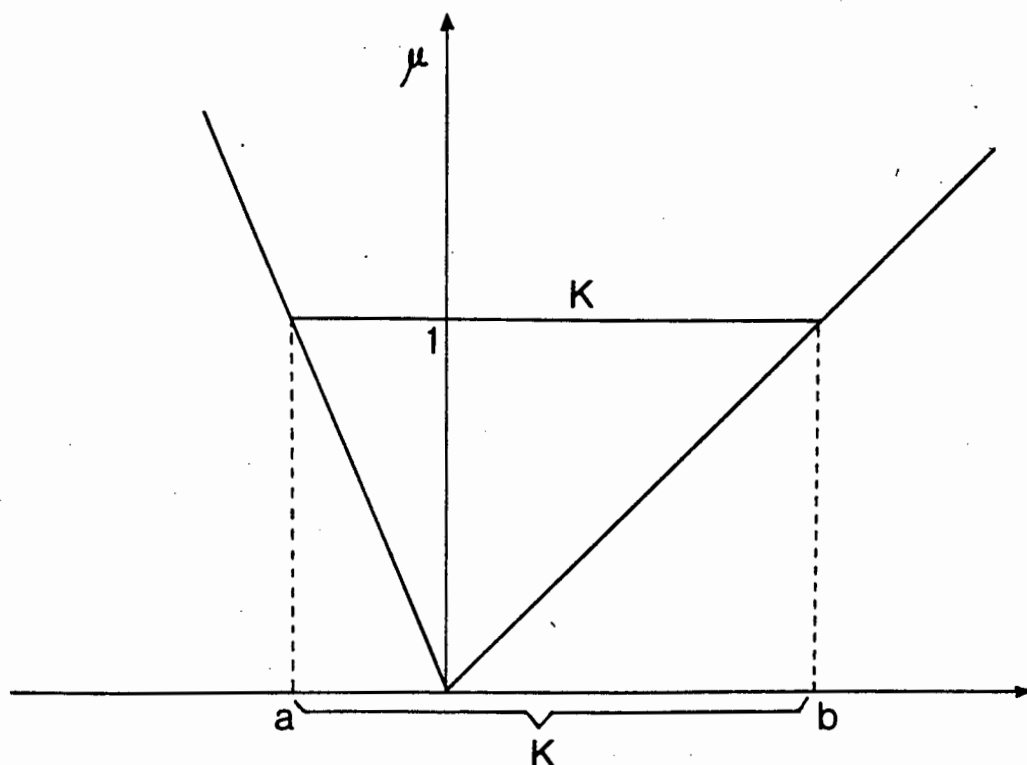


Figure 2.7 The gauge function of the set $K = [a, b]$.

LEMMA 2.1 Let K be a closed and convex set. Then

- (a) the support function D_K is a gauge;
- (b) $K = \partial D_K(0)$ (see Figure 2.8);
- (c) the subdifferential of the indicator function is the normal cone: $\partial I_K = N_K$;

(d) if $D_K(p) = x \cdot p$ (supremum achieved at x) then

$$x \in \partial D_K(p) \Leftrightarrow p \in \partial D_K^*(x) = N_K(x) \quad . \quad (2.17)$$

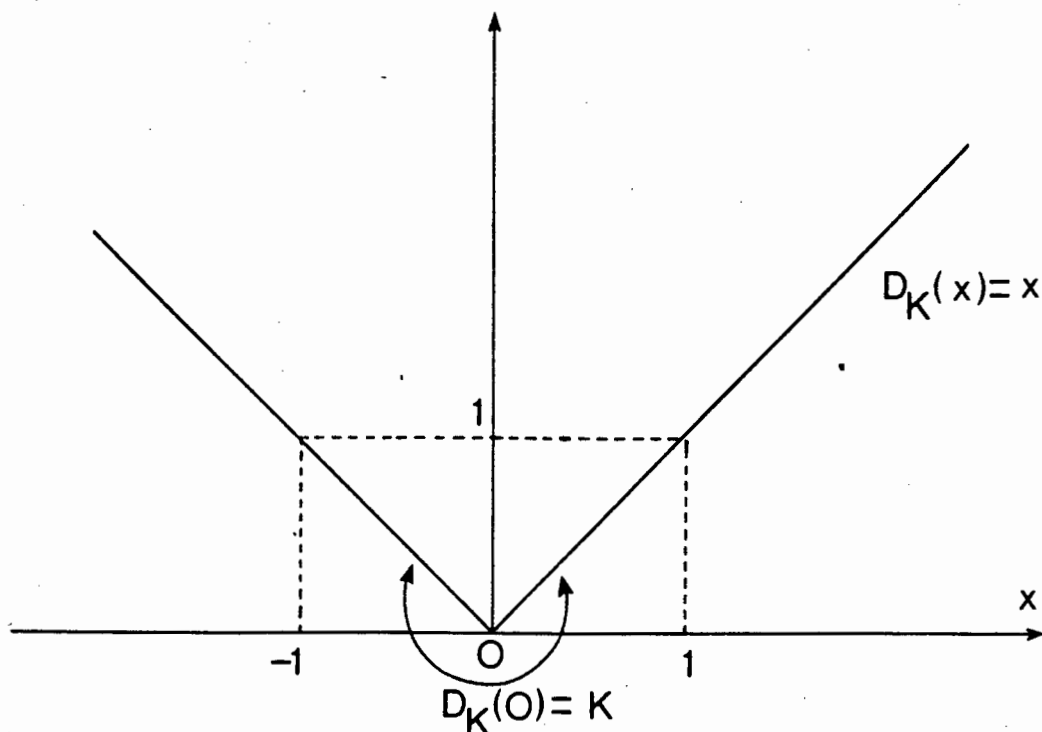


Figure 2.8 The subdifferential ∂D_K of the support function $D_K = x$ at the origin for $K = [-1, 1]$.

REMARK. Part (d) of the above LEMMA is a consequence of the fact that the support and indicator functions of a convex set are conjugate to each other, in the sense of Legendre-Fenchel [69].

The conjugate pairs (x, p) are illustrated in Figure 2.9 in relation to the convex regions K and C , where $K = \{x : f(x) \leq 1\}$ and C is the level set of $D_K = \text{constant}$. It is shown in the figure that the vector p belongs to the normal cone N_K of K at the points x, x' ,

etc., so that all these points are in the subdifferential ∂D_K of D_K at p . The vectors x' and x'' in K become the extreme elements in the normal cone at the boundary of C . Since from (2.6)

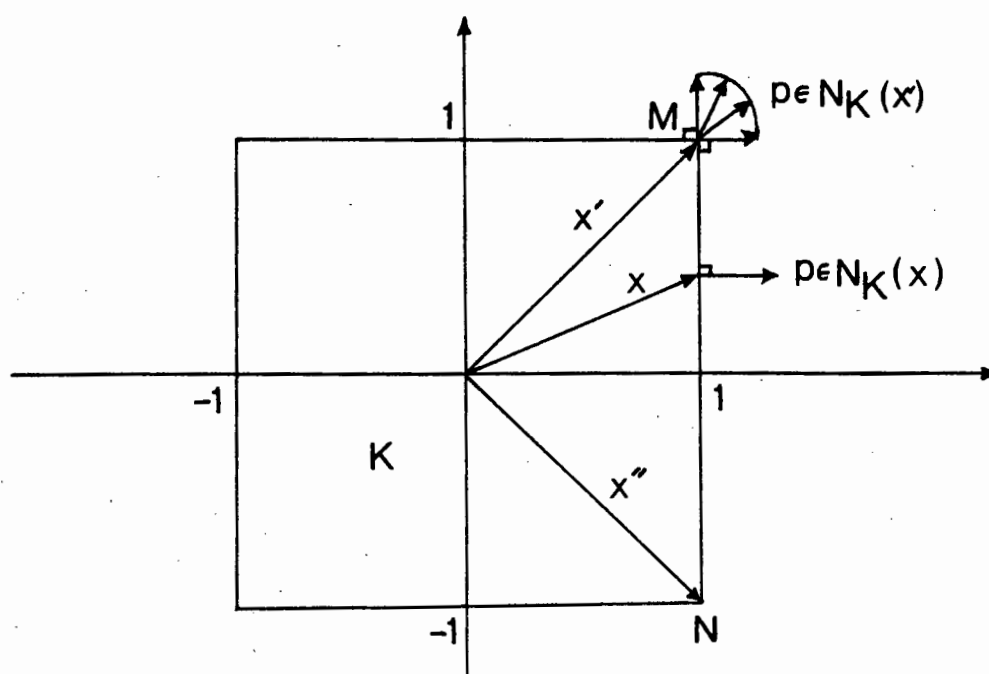
$$D_K(p) = \sup_{y \in K} y \cdot p = x \cdot p$$

we can see through trigonometry that for any p and x between x' and x'' the corresponding point in C becomes the corner A in the part (b) of Figure 2.9. Hence the line \bar{MN} in part (a) of the figure corresponds to the point A , and the point M corresponds to the line \bar{AB} in (b). Evidently Figure 2.9 is an illustration of the statement

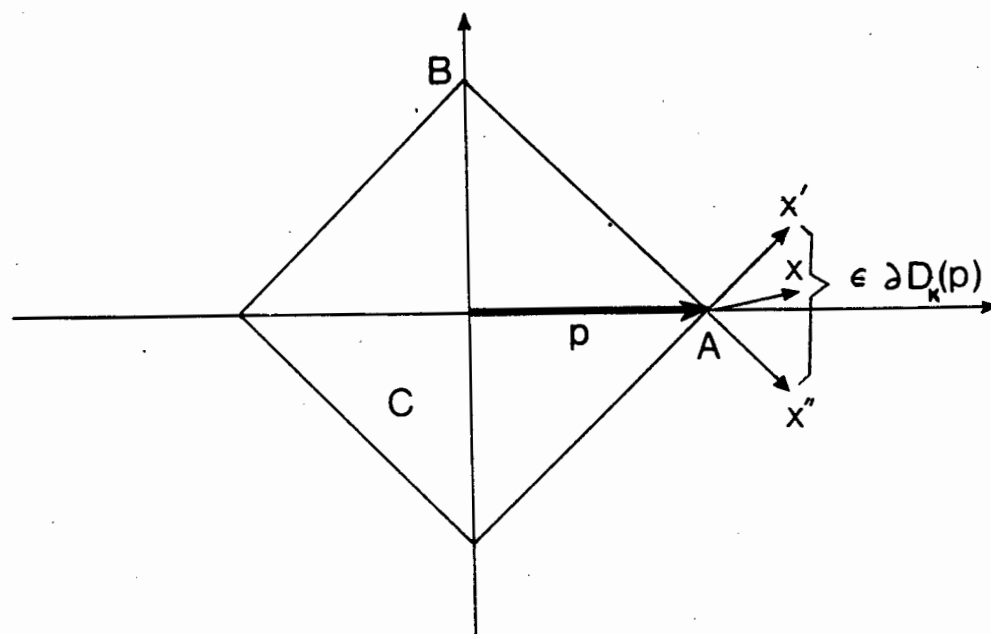
$$x \in \partial D_K(p) \quad \text{if and only if} \quad p \in N_K(x) \quad ,$$

that is part (d) of Lemma 2.1.

In the context of plasticity, the closed and convex set K is interpreted as the region of admissible stresses in which case the boundary of K becomes the yield surface. In the case of finite-strain plasticity p is interpreted as the rate of the *internal variable*, and the support function D_K is the *dissipation function* associated with p . Part (d) of Lemma 2.1 yields two alternate forms of the evolution equation, the second one being the normality law. The connection with a yield function is made in the following result [26].



(a) The set K and the normal at its boundary.



(b) The level curve $D = \text{constant}$.

Figure 2.9 The conjugate pair (x, p) in the set K in (a) and in C in (b) which is the level set $D_K = \text{constant}$.

LEMMA 2.2 Let D be a gauge with the property that

$$\mathbf{p} \neq \mathbf{0} \Rightarrow D(\mathbf{p}) \neq 0, \quad (2.18)$$

and with f defined by (2.14) where K is a closed and convex set, with $\mathbf{0} \in K$. Then

(a) f is a gauge, and

$$K = \{\mathbf{x} : f(\mathbf{x}) \leq 1\}; \quad (2.19)$$

(b) f and D are polar to each other in the following sense

$$f(\mathbf{x}) = \sup_{\mathbf{p} \neq \mathbf{0}} \frac{\mathbf{x} \cdot \mathbf{p}}{D(\mathbf{p})} \quad \text{and} \quad D(\mathbf{p}) = \sup_{\mathbf{x} \neq \mathbf{0}} \frac{\mathbf{x} \cdot \mathbf{p}}{f(\mathbf{x})};$$

(c) if $\mathbf{x} \in \partial D(\mathbf{p})$ then

$$\mathbf{x} \cdot \mathbf{p} = D(\mathbf{p})f(\mathbf{x});$$

(d) if $\mathbf{p} \neq \mathbf{0}$ and $\mathbf{x} \in \partial D(\mathbf{p})$ then there exists a non-negative scalar $\lambda \geq 0$ such that

$$\mathbf{p} \in \lambda \partial f(\mathbf{x}) \quad (2.20)$$

or, if f is smooth,

$$\mathbf{p} = \lambda \nabla f(\mathbf{x}) \equiv \lambda \frac{\partial f(\mathbf{x})}{\partial \mathbf{x}}. \quad (2.21)$$

If we interpret the results in the context of plasticity, we can immediately recognise f as the *yield function*. Although there are infinitely many functions $g(\mathbf{x})$, whose level surfaces $g(\mathbf{x}) = 1$

can be identified as the *yield surface*, the Lemma identifies one such function f which has the advantage of being a gauge; furthermore it is related to the dissipation function D . Therefore the function f is called the *canonical yield function*.

The construction of f from a convex set K is illustrated in Figure 2.10, where the set K is shown as a region on the (x_1, x_2) plane.

The *evolution equation* associated with any internal variable theory of plasticity provides information about the internal variable rate \mathbf{p} , generally in the form

$$\mathbf{p} = F(\dots) \quad (2.22)$$

where the function F and its arguments need to be specified.

By the use of Theorem 2.1 with Lemma 2.1 the evolution equation of the classical theory of plasticity can be formulated in three equivalent ways, which we briefly describe in the following form:

$$\mathbf{x} \in H(\mathbf{p}) \Leftrightarrow \mathbf{x} \in \partial D_K(\mathbf{p}) \Leftrightarrow \mathbf{p} \in N_K(\mathbf{x}) \quad ; \quad (2.23)$$

where H is maximal responsive; D_K is convex, positively homogeneous, lower semicontinuous; and K is closed and convex.

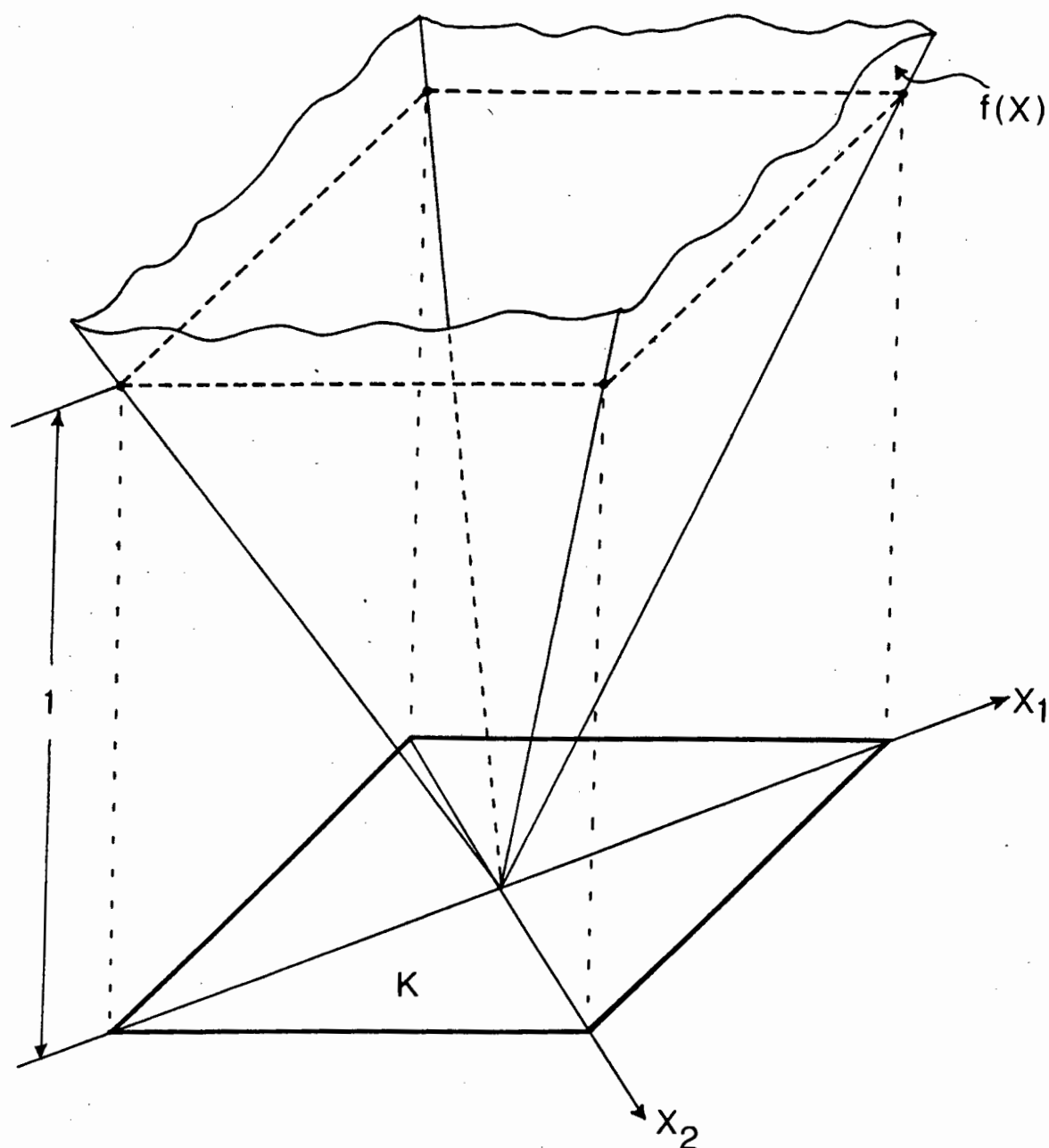


Figure 2.10 The construction of a canonical yield function f from the convex set K .

If we use Lemma 2.2 with the assumption (2.12) we can introduce the canonical yield function f which is the polar conjugate to the gauge D ; note that $D = D_K$ by definition and Lemma 2.1. Theorem 2.1 clearly indicates the equivalence of the maximum plastic work inequality to the notion of a convex yield surface and the normality law. We can easily identify the set of inequalities (2.5) with the *maximum plastic work* inequality, which will be discussed in the next Chapter.

CHAPTER III

CONSTITUTIVE THEORY OF FINITE STRAIN PLASTICITY

3.1 INTRODUCTION

In this Chapter we present an internal variable finite-strain theory of plasticity within the framework of convex analysis. This theory has since been published in [25], and is an extension of the theory of Eve, Reddy, Rockafellar [26] for small strain elasto-plasticity. We also incorporate further investigation and results in the case of non-smooth yield surfaces. The work in [25] was joint work done in collaboration with B D Reddy and R A Eve, but it has been written up independently. Certain kinematical results presented in Section 3.2 are the extensions of the author. Section 3.6 is the contribution of the author in the case of non-smooth yield surfaces; this does not appear in [26].

The theory is based on the assumption of the convexity of the *yield surface* and the *normality law*. These assumptions might seem to be rather restrictive in the sense that there are materials exhibiting plastic behaviour with non-convex yield surfaces, for example, and certain materials whose behaviour is more closely approximated by flow laws in which case the normality law is dropped. However, these assumptions enable us to obtain some useful and

general results, and to gain insight into the plastic behaviour of materials by using the methods of convex analysis. While remaining restricted to convex yield surfaces, the theory is not however restricted to non-smooth yield surfaces, and so includes as special cases surfaces such as that corresponding to the Tresca yield condition. In spite of the restriction of the convexity of the yield function, the theory provides a sufficiently broad framework within which a wide range of materials exhibiting elastic-plastic behaviour can be modelled.

We explain the kinematics of nonlinear elastic-plastic continua in Section 3.2 where we propose a multiplicative split of the total deformation gradient tensor into its elastic and plastic parts. We proceed with the description of conservation laws, thermodynamics and stress-strain relations in Section 3.3. We present the construction of the evolution equation and its relation to the yield function in Section 3.4. We take as an example Ciarlet's free energy function for nonlinear elastic materials and extend it to the case of elastic-plastic materials; this strain energy function is then used in various manipulations for the clarification of concepts, in Section 3.5. In Section 3.6 we present an extended version of [25] in which Sewell's [70,71] theory for non-smooth convex yield functions is incorporated into the theory presented in [25]. Although [25] covers non-smooth yield surfaces it is not convenient in its existing form for the study of singular surfaces in plastic media. Therefore, this theory has been extended to provide a link with Sewell's treatment of non-smooth yield surfaces, which is more convenient for the treatment

of singular surfaces. Sewell's theory is confined to materials exhibiting isotropic hardening, and we also consider only this form of hardening.

3.2 KINEMATICS OF NONLINEAR ELASTIC-PLASTIC CONTINUA

A body in the context of continuum mechanics is identified with a region B in some fixed reference configuration at a specified time t_0 , where B is a subset of the Euclidean point space E . The position of a material particle at time t is defined by the motion $\mathbf{x} = \mathbf{x}(\mathbf{X}, t)$. Here and henceforth we denote the function and its value by the same symbol. \mathbf{X} denotes the position vector of the particle in the *reference configuration* at time t_0 . We assume that the function \mathbf{x} is invertible, hence $\mathbf{X} = \mathbf{X}(\mathbf{x}, t)$ exists; and both \mathbf{x} and \mathbf{X} are twice continuously differentiable in the spatial and temporal variables on which they depend, unless otherwise specified.

Throughout this thesis the material (Lagrangian) and the spatial (Eulerian) coordinates will be defined with respect to a fixed set of axes with an orthonormal basis \mathbf{e}_k .

In a rectangular cartesian coordinate system the i -th component of \mathbf{x} is given by

$$x_i = x_i(X_1, X_2, X_3, t) \quad , \quad i = 1, 2, 3 \quad , \quad (3.1)$$

where X_1, X_2, X_3 are the coordinates of the material particle in the reference configuration.

The deformation gradient \mathbf{F} is defined by

$$\mathbf{F}(\mathbf{X}, t) \equiv \text{Grad } \mathbf{x} = \frac{\partial \mathbf{x}_k}{\partial X_L} \mathbf{e}_k \otimes \mathbf{e}_L \quad (3.2)$$

or

$$\mathbf{F} = F_{kL} \mathbf{e}_k \otimes \mathbf{e}_L$$

where indices referring to quantities defined in the current and reference configuration are respectively denoted by lower and upper case letters.

We assume that the determinant of the deformation gradient, that is the Jacobian J , exists at each point in the region B and that

$$J \equiv \det \mathbf{F} > 0 \quad (3.3)$$

The *physical implication* of (3.3) is that the material constituting the body cannot penetrate itself and that the material occupying any finite volume can neither be compressed to a point nor be expanded to an infinite volume by any motion [2]. The *mathematical implication* of (3.3) is the existence of a unique inverse \mathbf{F}^{-1} of \mathbf{F} [14], given by

$$\mathbf{F}^{-1} = F_{K\ell}^{-1} \mathbf{e}_K \otimes \mathbf{e}_\ell \quad (3.4)$$

where $F_{K\ell}^{-1}$ denotes the matrix of components of the inverse of F .
From (3.1) and (3.2)

$$F^{-1} = \frac{\partial X_K}{\partial x_\ell} e_K \otimes e_\ell \quad .$$

The total deformation in the context of elastic-plastic behaviour of materials is partly elastic and partly plastic. Distinction of elastic and plastic parts of the total deformation is necessary for describing the kinematics of such behaviour. Various researchers have adopted one basic method, in which the total strain is decomposed into its elastic and plastic parts; this was proposed by Green and Naghdi [27]. The second, and more commonly used, is the multiplicative decomposition of the deformation gradient tensor proposed by Lee [40]. Following Lee we will use the multiplicative split of the deformation gradient tensor into its elastic and plastic counterparts, and set

$$F = F^e F^p \tag{3.5}$$

where F^e and F^p denote, respectively, the elastic and plastic parts. We observe that F , F^e and F^p behave in the following way as operators:

$$\begin{aligned} F^p &: N(X) \longrightarrow N(I) \quad , \\ F^e &: N(I) \longrightarrow n(x) \quad , \end{aligned}$$

and

$$F : N(\mathbf{X}) \longrightarrow n(\mathbf{x}) \quad ,$$

where $N(\mathbf{X})$ denotes a neighbourhood of \mathbf{X} in the reference configuration, $N(\mathbf{x})$ the image of $N(\mathbf{X})$ in an intermediate configuration under F^p , and $n(\mathbf{x})$ the image of $N(\mathbf{X})$ in the current configuration under F . (See Figure 3.1.)

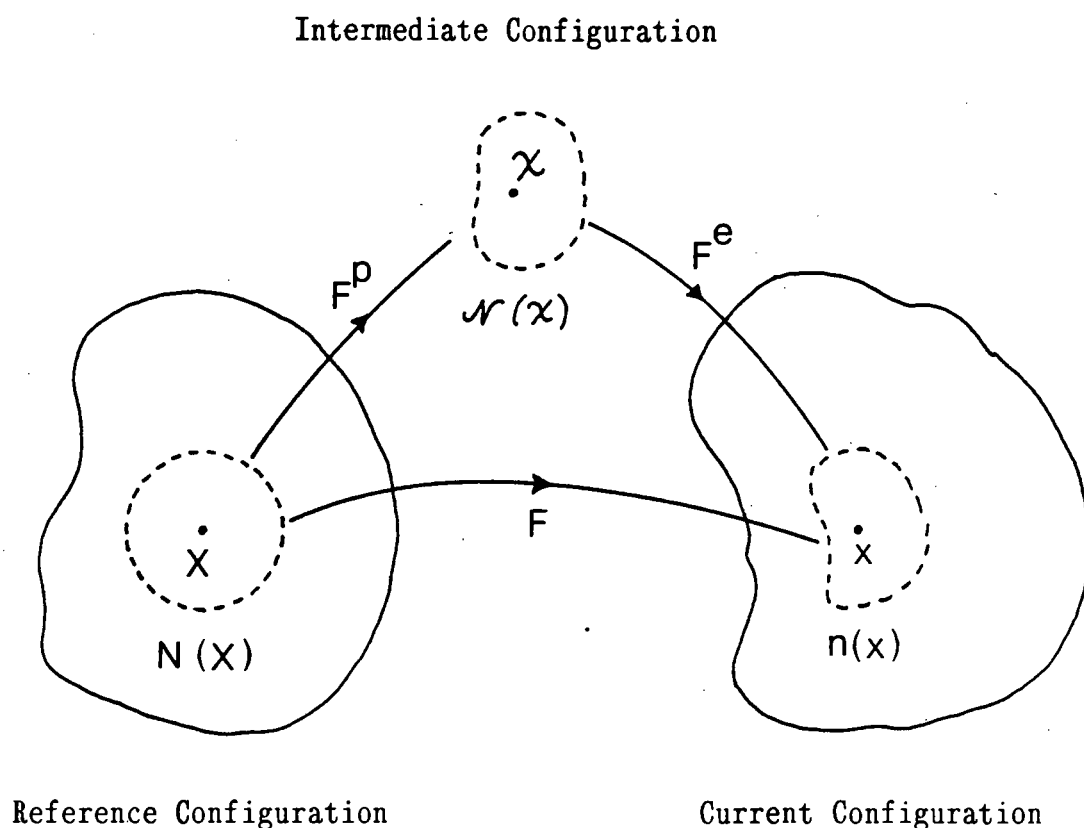


Figure 3.1 Reference and current configurations of a material body and the reference, intermediate and current configurations of a neighbourhood $N(\mathbf{X})$ of a material point \mathbf{X} .

We use lower case Greek letters to denote components associated with the intermediate configuration, so that (3.5) reads, in component form, $F_{kL} = F_{ka}^e F_{aL}^p$. F^p can be regarded as a local map of a

neighbourhood of a material point in the reference configuration which is solely due to plastic deformation. (See Figure 3.1) The intermediate configuration of a neighbourhood of a material point can alternatively be regarded as the configuration reached by such a neighbourhood after it has been mapped to its current configuration and then allowed to unload elastically. However \mathbf{F}^P is *not* the gradient of a motion from the reference to the intermediate configuration [40].

The abovementioned decomposition (3.5) is not unique since for any orthogonal tensor \mathbf{Q} we can write

$$\mathbf{F}^e \mathbf{F}^P = \mathbf{F}^e \mathbf{Q} \mathbf{Q}^T \mathbf{F}^P, \quad (3.6)$$

so that by setting $\mathbf{G}^e = \mathbf{F}^e \mathbf{Q}$ and $\mathbf{G}^P = \mathbf{Q}^T \mathbf{F}^P$ we have

$$\mathbf{F} = \mathbf{F}^e \mathbf{F}^P = \mathbf{G}^e \mathbf{G}^P,$$

where \mathbf{G}^e and \mathbf{G}^P become the alternative elastic and plastic parts of \mathbf{F} . For any specific application, though, a unique decomposition can be chosen by making a specific choice of \mathbf{Q} .

By using the polar decomposition theorem \mathbf{F} , \mathbf{F}^e and \mathbf{F}^P can be decomposed uniquely in the forms

$$\begin{aligned} \mathbf{F} &= \mathbf{R} \mathbf{U} = \mathbf{V} \mathbf{R} \quad , \quad \mathbf{F}^e = \mathbf{R}^e \mathbf{U}^e = \mathbf{V}^e \mathbf{R}^e \quad , \\ \mathbf{F}^P &= \mathbf{R}^P \mathbf{U}^P = \mathbf{V}^P \mathbf{R}^P \quad , \end{aligned} \quad (3.7)$$

where U , V , U^e , V^e , U^p and V^p are positive definite symmetric tensors and R , R^e , R^p are proper orthogonal tensors. The tensors U and V represent the total stretches, U^e and V^e represent the elastic stretches, and U^p and V^p represent the plastic stretches; R , R^e and R^p represent total, elastic and plastic rotations respectively. The decomposition (3.5) can be made unique by choosing either $R^e = I$, so that F^e is a positive definite symmetric tensor, in which case all of the rotation is contained in F^p , or $R^p = I$, so that F^p is a positive definite symmetric tensor and all the rotation is contained in F^e ; that is

$$F = V^e F^e \quad \text{or} \quad F = F^e U^p \quad . \quad (3.8)$$

Although we do not specify any particular choice of decomposition at this stage, we assume that a decomposition has been chosen once and for all. Since F^p is not the gradient of a motion from the reference to the intermediate configuration it should be noted that the above interpretations only apply locally.

The right Cauchy-Green strain tensor is defined by

$$C = F^T F \quad . \quad (3.9)$$

An elastic counterpart C^e of C and a plastic counterpart C^p may be defined respectively by

$$C^e = F^{eT} F^e \quad (3.10)$$

and

$$\mathbf{C}^p = \mathbf{F}^p \mathbf{F}^{pT} \quad . \quad (3.11)$$

The left Cauchy-Green strain tensor is defined by

$$\mathbf{B} = \mathbf{F} \mathbf{F}^T \quad . \quad (3.12)$$

Elastic and plastic counterparts of \mathbf{B} may similarly be defined by

$$\mathbf{B}^e = \mathbf{F}^e \mathbf{F}^{eT} \quad (3.13)$$

and

$$\mathbf{B}^p = \mathbf{F}^p \mathbf{F}^{pT} \quad . \quad (3.14)$$

The Cauchy-Green strain tensors (which are positive-definite and symmetric) have the spectral representations

$$\mathbf{C} = \sum_{i=1}^3 c_i \mathbf{p}_i \otimes \mathbf{p}_i \quad , \quad (3.15)$$

$$\mathbf{B} = \sum_{i=1}^3 c_i \mathbf{q}_i \otimes \mathbf{q}_i \quad , \quad (3.16)$$

where $c_i > 0$ are the proper numbers of \mathbf{C} and \mathbf{B} ; the triads $\{\mathbf{p}_i\}$ and $\{\mathbf{q}_i\}$ define locally two sets of orthonormal principal axes, respectively, in reference and current configurations, that are the proper vectors of \mathbf{C} and \mathbf{B} . The scalars $a_i = \sqrt{c_i}$ are called the *principal stretches*. The two sets of principal axes in the reference and

current configurations are related by

$$\mathbf{q}_i = \mathbf{R} \mathbf{p}_i \quad (3.17)$$

where the rotation tensor \mathbf{R} is given by

$$\mathbf{R} = \sum_{i=1}^3 \mathbf{q}_i \otimes \mathbf{p}_i \quad (3.18)$$

From (3.7) we get the principal representations

$$\mathbf{U}^e = \sum_{i=1}^3 a_i^e \beta_i \otimes \beta_i, \quad (3.19)$$

$$\mathbf{V}^e = \sum_{i=1}^3 a_i^e \mathbf{q}_i \otimes \mathbf{q}_i \quad (3.20)$$

where the triad $\{\beta_i\}$ denotes the proper vectors of \mathbf{U}^e , that is the principal axes in the intermediate configuration, and a_i^e denotes the proper numbers of both \mathbf{U}^e and \mathbf{V}^e which we call the *elastic principal stretches*. The principal axes in the current and intermediate configurations are related by

$$\mathbf{q}_i = \mathbf{R}^e \beta_i \quad (3.21)$$

where the rotation tensor \mathbf{R}^e is

$$\mathbf{R}^e = \sum_{i=1}^3 \mathbf{q}_i \otimes \beta_i \quad (3.22)$$

Similarly from (3.8) we get the expressions

$$\mathbf{U}^p = \sum_{i=1}^3 a_i^p \mathbf{p}_i \otimes \mathbf{p}_i, \quad (3.23)$$

$$\mathbf{V}^p = \sum_{i=1}^3 a_i^p \beta_i \otimes \beta_i, \quad (3.24)$$

$$\beta_i = \mathbf{R}^p \mathbf{p}_i \quad (3.25)$$

where a_i^p are the proper numbers of \mathbf{U}^p and \mathbf{V}^p which we call the *plastic principal stretches* and the rotation tensor \mathbf{R}^p is

$$\mathbf{R}^p = \sum_{i=1}^3 \beta_i \otimes \mathbf{p}_i. \quad (3.26)$$

From (3.7), (3.10) and (3.13) we find that the elastic counterparts \mathbf{C}^e and \mathbf{B}^e of \mathbf{C} and \mathbf{B} are given by

$$\mathbf{C}^e = \sum_{i=1}^3 (a_i^e)^2 \beta_i \otimes \beta_i, \quad (3.27)$$

$$\mathbf{B}^e = \sum_{i=1}^3 (a_i^e)^2 \mathbf{q}_i \otimes \mathbf{q}_i. \quad (3.28)$$

Similarly the plastic counterparts \mathbf{C}^p and \mathbf{B}^p of \mathbf{C} and \mathbf{B} are, in spectral form

$$\mathbf{C}^p = \sum_{i=1}^3 (a_i^p)^2 \mathbf{p}_i \otimes \mathbf{p}_i \quad , \quad (3.29)$$

$$\mathbf{B}^p = \sum_{i=1}^3 (a_i^p)^2 \boldsymbol{\beta}_i \otimes \boldsymbol{\beta}_i \quad . \quad (3.30)$$

Equations (3.7), (3.19) and (3.21) give the elastic counterpart \mathbf{F}^e of \mathbf{F} :

$$\mathbf{F}^e = \sum_{i=1}^3 a_i^e \mathbf{q}_i \otimes \boldsymbol{\beta}_i \quad . \quad (3.31)$$

Similarly the plastic counterpart \mathbf{F}^p of \mathbf{F} is

$$\mathbf{F}^p = \sum_{i=1}^3 a_i^p \boldsymbol{\beta}_i \otimes \mathbf{p}_i \quad . \quad (3.32)$$

If we substitute (3.31) and (3.32) in (3.5) we get

$$\mathbf{F} = \sum_{i=1}^3 a_i \mathbf{q}_i \otimes \mathbf{p}_i \quad (3.33)$$

where $a_i = a_i^e a_i^p$ (no sum) gives the multiplicative split of the principal stretches into their elastic and plastic parts.

The velocity gradient \mathbf{L} is related to the rate of change of deformation gradient by

$$\mathbf{L} = \dot{\mathbf{F}}\mathbf{F}^{-1}, \quad (3.34)$$

and may be decomposed into its elastic and plastic parts by using (3.5) and (3.34), to give

$$\mathbf{L} = \dot{\mathbf{F}}^e \mathbf{F}^p \mathbf{F}^{-1} + \mathbf{F}^e \dot{\mathbf{F}}^p \mathbf{F}^{-1} \quad (3.35)$$

or $\mathbf{L} = \mathbf{L}^e + \mathbf{L}^p$ where

$$\mathbf{L}^e = \dot{\mathbf{F}}^e \mathbf{F}^p \mathbf{F}^{-1} \quad (3.36)$$

and

$$\mathbf{L}^p = \mathbf{F}^e \dot{\mathbf{F}}^p \mathbf{F}^{-1}. \quad (3.37)$$

The deformation rate \mathbf{D} is defined by

$$\mathbf{D} = \frac{1}{2}(\mathbf{L} + \mathbf{L}^T) \quad (3.38)$$

and may be decomposed into its elastic and plastic parts \mathbf{D}^e and \mathbf{D}^p by using (3.35) and (3.38), so that

$$\mathbf{D} = \mathbf{D}^e + \mathbf{D}^p \quad (3.39)$$

where

$$\mathbf{D}^e = \frac{1}{2}(\mathbf{L}^e + \mathbf{L}^{eT}) \quad (3.40)$$

The velocity gradient \mathbf{L} is related to the rate of change of deformation gradient by

$$\mathbf{L} = \dot{\mathbf{F}}\mathbf{F}^{-1}, \quad (3.34)$$

and may be decomposed into its elastic and plastic parts by using (3.5) and (3.34), to give

$$\mathbf{L} = \dot{\mathbf{F}}^e \mathbf{F}^p \mathbf{F}^{-1} + \mathbf{F}^e \dot{\mathbf{F}}^p \mathbf{F}^{-1} \quad (3.35)$$

or $\mathbf{L} = \mathbf{L}^e + \mathbf{L}^p$ where

$$\mathbf{L}^e = \dot{\mathbf{F}}^e \mathbf{F}^p \mathbf{F}^{-1} \quad (3.36)$$

and

$$\mathbf{L}^p = \mathbf{F}^e \dot{\mathbf{F}}^p \mathbf{F}^{-1}. \quad (3.37)$$

The deformation rate \mathbf{D} is defined by

$$\mathbf{D} = \frac{1}{2}(\mathbf{L} + \mathbf{L}^T) \quad (3.38)$$

and may be decomposed into its elastic and plastic parts \mathbf{D}^e and \mathbf{D}^p by using (3.35) and (3.38), so that

$$\mathbf{D} = \mathbf{D}^e + \mathbf{D}^p \quad (3.39)$$

where

$$\mathbf{D}^e = \frac{1}{2}(\mathbf{L}^e + \mathbf{L}^{eT}) \quad (3.40)$$

and

$$\mathbf{D}^p = \frac{1}{2}(\mathbf{L}^p + \mathbf{L}^{pT}) \quad . \quad (3.41)$$

From (3.10) we obtain the time derivative of \mathbf{C}^e :

$$\dot{\mathbf{C}}^e = \dot{\mathbf{F}}^{eT} \mathbf{F}^e + \mathbf{F}^{eT} \dot{\mathbf{F}}^e \quad . \quad (3.42)$$

From (3.36) and (3.5) we have

$$\dot{\mathbf{F}}^e = \mathbf{L}^e \mathbf{F}^e \quad (3.43)$$

which we substitute in (3.42) to get

$$\dot{\mathbf{C}}^e = \mathbf{F}^{eT} (\mathbf{L}^{eT} + \mathbf{L}^e) \mathbf{F}^e = 2\mathbf{F}^{eT} \mathbf{D}^e \mathbf{F}^e \quad . \quad (3.44)$$

A common assumption of the kinematics of plastic materials is that there is no volumetric change accompanying plastic deformation. This constraint can be imposed by setting

$$\det \mathbf{F}^p = 1 \quad , \quad (3.45)$$

which in turn implies that

$$0 = (\det \mathbf{F}^p)^{\cdot} = (\det \mathbf{F}^p) \operatorname{tr} \mathbf{L}^p \quad (3.46)$$

or, if we make the physically reasonable assumption that $\det \mathbf{F}^p > 0$, we conclude that

$$\text{tr } \mathbf{L}^p = 0 \quad (3.47)$$

or, equivalently,

$$\text{tr } \mathbf{D}^p = 0 \quad (3.48)$$

Here \det and tr denote the determinant and trace of a matrix or second-rank tensor.

We finally define the *spin tensor* \mathbf{W} by

$$\mathbf{W} = \frac{1}{2}(\mathbf{L} - \mathbf{L}^T) \quad , \quad (3.48a)$$

and can be decomposed into its elastic and plastic parts \mathbf{W}^e and \mathbf{W}^p by

$$\mathbf{W} = \mathbf{W}^e + \mathbf{W}^p \quad (3.48b)$$

Similar to (3.40) and (3.41) \mathbf{W}^e and \mathbf{W}^p are defined as follows

$$\mathbf{W}^e = \frac{1}{2}(\mathbf{L}^e - \mathbf{L}^{eT}) \quad , \quad (3.48c)$$

$$\mathbf{W}^p = \frac{1}{2}(\mathbf{L}^p - \mathbf{L}^{pT}) \quad .$$

3.3 BALANCE LAWS, THERMODYNAMICS AND STRESS-STRAIN RELATIONS

We define here various stress tensors which will be of use later. The *Cauchy* or *true stress* σ is defined in the current configuration of the material body, is symmetric and has the representation $\sigma = \sigma_{ij} \mathbf{e}_i \otimes \mathbf{e}_j$. A related measure of stress is the *Kirchhoff stress tensor* $\bar{\sigma}$ which is related to the Cauchy stress by

$$\bar{\sigma} = J\sigma \quad . \quad (3.49)$$

The *first Piola-Kirchhoff stress tensor* \mathbf{T} is related to σ by

$$\mathbf{T} = J\sigma\mathbf{F}^{-T} \quad , \quad (3.50)$$

and its components are interpretable as forces per unit reference area through the identity $\sigma \mathbf{n} \, da = \mathbf{T} \mathbf{N} \, dA$. Here da and dA denote, respectively, the differential area elements in current and reference configurations, and \mathbf{n} and \mathbf{N} the current and referential unit vectors. This stress is represented in an orthonormal basis by $\mathbf{T} = T_{kL} \mathbf{e}_k \otimes \mathbf{e}_L$. The symmetric second *Piola-Kirchhoff stress* \mathbf{S} is defined by

$$\mathbf{S} = \mathbf{T}^T \mathbf{F}^{-T} = J\mathbf{F}^{-1} \sigma \mathbf{F}^{-T} \quad , \quad (3.51)$$

and has the representation $\mathbf{S} = S_{KL} \mathbf{e}_K \otimes \mathbf{e}_L$; thus it is associated with the reference configuration. The usefulness of defining these different stress measures can be seen from the expression for specific rate of work, which is given by $\bar{\sigma} \cdot \mathbf{D} = \mathbf{T} \cdot \dot{\mathbf{F}} = \frac{1}{2} \mathbf{S} \cdot \dot{\mathbf{C}}$. Here

$\mathbf{A} \cdot \mathbf{B} = A_{ij} B_{ij}$, for example, defines the scalar product of two tensors \mathbf{A} and \mathbf{B} .

In deformable media the local forms of the laws of balance of *linear momentum* and *angular momentum* written in the reference configuration are, respectively,

$$\text{Div } \mathbf{T} + \rho_0 \mathbf{b} = \rho_0 \ddot{\mathbf{x}} \quad (3.52)$$

and

$$\mathbf{T} \mathbf{F}^T = \mathbf{F} \mathbf{T}^T \quad , \quad (3.53)$$

where \mathbf{b} is the specific body force, $\ddot{\mathbf{x}}$ is the acceleration, ρ_0 the mass density in the reference configuration and \mathbf{F} the deformation gradient as usual. The operator Div is the divergence in the reference configuration, and is defined by

$$\text{Div } \mathbf{T} = \frac{\partial T_{iA}}{\partial X_A} \mathbf{e}_i \quad .$$

The equation of *balance of energy* for deformable media is

$$\rho_0 (\dot{\psi} + s\dot{\theta}) = \mathbf{T} \cdot \dot{\mathbf{F}} - \text{Div } \mathbf{q} + \rho_0 r \quad (3.54)$$

and the *second law of thermodynamics* (Clausius-Duhem inequality) is

$$\rho_0 (\dot{\psi} + s\dot{\theta}) - \mathbf{T} \cdot \dot{\mathbf{F}} - \theta^{-1} \mathbf{q} \cdot \text{Grad } \theta \leq 0 \quad (3.55)$$

where ψ is the free energy, θ temperature, s the entropy density, \mathbf{q} the referential heat flux vector and r the heat supply [19]. A superposed dot here denotes material differentiation with respect to time. $\text{Div } \mathbf{q}$ denotes the divergence operator on \mathbf{q} in reference coordinates, that is $\text{Div } \mathbf{q} = \partial q_K / \partial X_K$. Also, $\text{Grad } \theta = (\partial \theta / \partial X_K) \mathbf{e}_K$.

In all of the above balance equations and in the inequality (3.55) we find it convenient to use the first Piola-Kirchhoff stress. In terms of the Cauchy stress the equations of the balance of linear and angular momenta (3.52) and (3.53) are respectively

$$\text{div } \boldsymbol{\sigma} + \rho \mathbf{b} = \rho_0 \ddot{\mathbf{x}} \quad , \quad (3.56)$$

$$\boldsymbol{\sigma} = \boldsymbol{\sigma}^T \quad , \quad (3.57)$$

and are known as Cauchy's equations of motion; here

$$\text{div } \boldsymbol{\sigma} = \frac{\partial \sigma_{ij}}{\partial x_j} \mathbf{e}_i \quad .$$

If we substitute (3.50) in (3.54) we can express the equation of balance of energy in the form

$$\rho_0 (\psi + s\theta)^\cdot = J(\boldsymbol{\sigma} \mathbf{F}^{-T}) \cdot \dot{\mathbf{F}} - \text{Div } \mathbf{q} + \rho_0 r \quad . \quad (3.58)$$

Since $\mathbf{T} \cdot \dot{\mathbf{F}} = J(\boldsymbol{\sigma} \mathbf{F}^{-T}) \cdot \dot{\mathbf{F}} = J(\boldsymbol{\sigma} \cdot \dot{\mathbf{F}} \mathbf{F}^{-1})$ and from (3.34) $\dot{\mathbf{F}} = \mathbf{L} \mathbf{F}$ we get $(\boldsymbol{\sigma} \mathbf{F}^{-T}) \cdot \dot{\mathbf{F}} = \boldsymbol{\sigma} \cdot \mathbf{L}$, but $\boldsymbol{\sigma}$ is a symmetric tensor, hence $\boldsymbol{\sigma} \cdot \mathbf{L} = \boldsymbol{\sigma} \cdot \frac{1}{2}(\mathbf{L} + \mathbf{L}^T)$. We recognise here the deformation rate \mathbf{D} from (3.38), therefore $\mathbf{T} \cdot \dot{\mathbf{F}} =$

$J\sigma \cdot D = \bar{\sigma} \cdot D$ and (3.58) becomes

$$\rho_0(\dot{\psi} + s\dot{\theta}) = \bar{\sigma} \cdot D - \text{Div } \mathbf{q} + \rho_0 r \quad (3.59)$$

while the Clausius-Duhem inequality (3.55) becomes

$$\rho_0(\dot{\psi} + s\dot{\theta}) - \bar{\sigma} \cdot D + \theta^{-1} \mathbf{q} \cdot \text{Grad } \theta \leq 0 \quad (3.60)$$

We will confine our attention to *isothermal* behaviour since we wish to focus on the mechanical aspects of inelastic behaviour of materials. The immediate result of this assumption is the considerable simplification of the equation of the balance of energy and the Clausius-Duhem inequality, because $\text{Grad } \theta = 0$, $\mathbf{q} = 0$ and $r = 0$, and θ is omitted as a variable.

We introduce a set of internal variables $\{\gamma, \mathbf{A}\}$ which describe hardening behaviour and other effects associated with internal rearrangements of the materials. The variable γ is a scalar while \mathbf{A} is a second rank symmetric tensor with respect to the reference configuration. We do not consider here internal variables which are vectors since it is known [19] that they would lead to a violation of the principle of *material frame indifference* which is an essential axiom of constitutive theories. In general there might be an array of both scalar and tensor types of internal variables, although we include only one of each kind in the development of the theory, for the sake of simplicity. The extension to arrays of internal variables will be a straightforward task.

We define, relative to the *intermediate configuration*, a symmetric "second Piola-Kirchhoff" stress by

$$\mathbf{S}^e = (\mathbf{F}^e)^{-1} \bar{\sigma}(\mathbf{F}^e)^{-T} . \quad (3.61)$$

Following the standard procedure as in Coleman and Gurtin [19] we consider a set of constitutive relations of the form in which

$$\psi = \psi(\mathbf{F}^e, \gamma, \mathbf{A}) \quad , \quad (3.62)$$

$$\mathbf{S}^e = \mathbf{S}^e(\mathbf{F}^e, \gamma, \mathbf{A}) \quad ,$$

together with an *evolution equation* which describes the evolution of inelastic behaviour. Instead of setting out explicitly the form which these equations take, we simply state at this stage that we seek a set of equations in which

$$\left\{ \begin{array}{c} \dot{\mathbf{F}}^D \\ \dot{\gamma} \\ \dot{\mathbf{A}} \end{array} \right\} \text{ are each functions of } \{\mathbf{F}^e, \gamma, \mathbf{A}\} \quad . \quad (3.63)$$

Rather than assuming a particular structure for the evolution equation at this stage, we will deduce an appropriate form from a reduced form of the dissipation inequality (Clausius-Duhem inequality) together with the assumptions based on convexity of certain sets and functions.

In the small-strain theory of elastoplasticity developed in [26], the free energy and stress are taken to be functions of *total* strain and internal variables. Although in most respects we will follow this theory, it proves to be more judicious in the case of finite strains to assume, as we have done above, that the free energy and stress depend on the *elastic* part of the deformation gradient \mathbf{F}^e as well as the internal variables. The reason for the viability of the small-strain theory with total strain as a variable is the linearity which pervades this theory, especially in the elastic part of the constitutive law. It is found that a finite-strain counterpart to this theory which makes use of total deformation (through \mathbf{F}) is unduly restrictive in its range of applicability.

We will apply the principle of material frame indifference to the set of constitutive equations for ψ and \mathbf{S}^e . A change of frame manifests itself as the transformation $\mathbf{F} \rightarrow \mathbf{QF}$, where \mathbf{Q} is a time-dependent proper orthogonal tensor. From the decomposition (3.5) it is clear that this implies the transformation $\mathbf{F}^e \rightarrow \mathbf{QF}^e$, the counterpart \mathbf{F}^p remaining unchanged by the change of frame. Similarly both internal variables remain unchanged by the transformation, γ because it is a scalar and \mathbf{A} because it is a referential quantity. The principle of material frame indifference precisely states that

$$\begin{aligned}\psi(\mathbf{F}^e, \gamma, \mathbf{A}) &= \psi(\mathbf{QF}^e, \gamma, \mathbf{A}) \quad , \\ \mathbf{S}^e(\mathbf{F}^e, \gamma, \mathbf{A}) &= \mathbf{S}^e(\mathbf{QF}^e, \gamma, \mathbf{A}) \quad .\end{aligned}$$

The above statement is an indication of the invariance of ψ and S^e between two observers which are mathematically characterized by two equivalent frames, that is, a point in one frame (\mathbf{x}, t) and a point in the other frame (\mathbf{x}', t) which are related by

$$\begin{aligned}\mathbf{x}' &= \mathbf{c}(t) + \mathbf{Q}(t)\mathbf{x} \quad , \\ t' &= t - a \quad ,\end{aligned}$$

where a is a constant, t is time and $\mathbf{Q}(t)$ is orthogonal.

Putting $\hat{\mathbf{F}}^e = \mathbf{Q}\mathbf{F}^e$ and using (3.10) we see that

$$\mathbf{C}^e = \mathbf{F}^{eT} \mathbf{F}^e = \mathbf{F}^{eT} \mathbf{Q}^T \mathbf{Q} \mathbf{F}^e = \hat{\mathbf{F}}^{eT} \hat{\mathbf{F}}^e$$

and furthermore from (3.7) that

$$\mathbf{C}^e = \mathbf{F}^{eT} \mathbf{F}^e = \mathbf{U}^e \mathbf{R}^{eT} \mathbf{R}^e \mathbf{U}^e = (\mathbf{U}^e)^2$$

which implies that

$$\mathbf{U}^e = (\mathbf{C}^e)^{\frac{1}{2}} \quad .$$

Thus, through the principle of material frame indifference we arrive at the reduced forms of the representations of ψ and S^e :

$$\psi = \psi(\mathbf{U}^e, \gamma, \mathbf{A}) \quad ,$$

(3.64)

$$S^e = S^e(\mathbf{U}^e, \gamma, \mathbf{A}) \quad ,$$

or

$$\begin{aligned}\psi &= \psi(\mathbf{C}^e, \gamma, \mathbf{A}) \quad , \\ S^e &= S^e(\mathbf{C}^e, \gamma, \mathbf{A}) \quad .\end{aligned}\tag{3.65}$$

We may define thermodynamic forces g and \mathbf{G} conjugate to γ and \mathbf{A} , respectively, by

$$\begin{aligned}g &= -\rho_0 \frac{\partial \psi}{\partial \gamma} \quad , \\ \mathbf{G} &= -\rho_0 \frac{\partial \psi}{\partial \mathbf{A}} \quad .\end{aligned}\tag{3.66}$$

The dissipation inequality

$$\rho_0 \dot{\psi} - \bar{\sigma} \cdot \mathbf{D} \leq 0 \tag{3.67}$$

is obtained from (3.55), and if we substitute (3.65) and (3.66) in (3.67) we get

$$\rho_0 \frac{\partial \psi}{\partial \mathbf{C}^e} \cdot \dot{\mathbf{C}}^e - g \dot{\gamma} - \mathbf{G} \cdot \dot{\mathbf{A}} - \bar{\sigma} \cdot \mathbf{D} \leq 0 \quad . \tag{3.68}$$

Using (3.39), (3.44) and noting that (3.68) must hold for all values of $\dot{\mathbf{C}}^e, \dot{\gamma}$ and $\dot{\mathbf{A}}$ we obtain the expression

$$S^e = 2\rho_0 \frac{\partial \psi}{\partial \mathbf{C}^e} \tag{3.69}$$

by splitting the elastic part of the dissipation inequality whence we obtain

$$\bar{\sigma} = 2\rho_0 \mathbf{F}^e \frac{\partial \psi}{\partial \mathbf{C}^e} \mathbf{F}^{eT} \quad (3.70)$$

for the Kirchhoff stress. With the assumption that *there is no plastic volume change* the reduced dissipation inequality becomes

$$g\dot{\gamma} + \mathbf{G} \cdot \dot{\mathbf{A}} + \text{dev } \bar{\sigma} \cdot \mathbf{D}^P \geq 0 \quad , \quad (3.71)$$

since $\mathbf{R} \cdot \mathbf{P} = \text{dev } \mathbf{R} \cdot \mathbf{P}$ if $\text{tr } \mathbf{P} = 0$; $\mathbf{R} = \text{dev } \mathbf{R} + \text{sph } \mathbf{R}$ where $\text{dev } \mathbf{R} = \mathbf{R} - \frac{1}{3}(\text{tr } \mathbf{R})\mathbf{I}$ is the deviatoric and $\text{sph } \mathbf{R}$ the spherical part of \mathbf{R} , for any tensor \mathbf{R} .

The reduced form (3.71) of the dissipation inequality which is obtained from the second law of thermodynamics (3.55) or (3.60) will be seen to play a pivotal role at a later stage in the construction of an evolution law.

3.4 THE EVOLUTION EQUATION AND THE YIELD FUNCTION

In this section we aim to resolve the issue of the form to be taken by the evolution equation whose general form was given in Chapter II by (2.22). The intention is to incorporate into this model the classical properties of convexity of yield surfaces and the normality law.

We defined in Chapter II a closed and convex set K which could be interpreted as the region of all admissible stresses, that is, the achievable values of \mathbf{X} . The boundary $\text{bd } K$ of K is known as the yield surface, while its interior $\text{int } K$ is commonly known as the *elastic region* since

$$\mathbf{X} \in \text{int } K \Rightarrow N_K(\mathbf{X}) = \{0\} \quad ,$$

that is,

$$p = 0 \quad \text{for } \mathbf{X} \in \text{int } K \quad . \quad (3.72)$$

The *yield surface* was defined by (2.19) in relation to the *yield function* $f(\mathbf{X})$. The values of \mathbf{X} which lie on $\text{bd } K$ or the yield surface are those which are associated with non-zero values of p . Therefore, we have

$$\mathbf{X} \in \text{bd } K \Rightarrow p \in N_K(\mathbf{X}) \neq \{0\} \quad , \quad (3.73)$$

which is the *normality rule*.

The groundwork for formalizing the ideas of the convexity of yield surfaces and the normality rule, has been laid in the previous Chapter. In order to develop a theory of plasticity in a convex analytic framework it is necessary first of all to decide on the configuration in which to postulate the existence of the appropriate convex sets of functions. Here we are guided by the important

observation [31] that, for metal plasticity at any rate, it is the yield function *as a function of Cauchy stress* which is observed to be convex. It is this notion of convexity which we will incorporate into the structure of the evolution equation, though we will postulate convexity with respect to Kirchhoff stress: since the Kirchhoff and Cauchy stresses differ by a scalar, any set or function which is convex with Cauchy stress as the variable, remains convex with Kirchhoff stress as the variable.

In order to ensure consistency in the arguments which follow, it is necessary that all quantities appearing in the evolution law (2.22) be spatial in character. The relevant physical quantities are those appearing in the reduced dissipation inequality (3.71). Of these, g and γ are scalars, and hence insensitive to the particular description, while $\bar{\sigma}$ and \mathbf{D}^p are spatial quantities. The manner in which the remaining two quantities \mathbf{G} and \mathbf{A} , are transformed into spatial ones will depend on their specific interpretations; for example, \mathbf{A} might represent a stress-type or a deformation-type quantity, and its spatial counterpart will differ accordingly. Rather than be unduly prescriptive, we assume that spatial counterparts $\mathbf{\hat{A}}$ and $\mathbf{\hat{G}}$ of \mathbf{A} and \mathbf{G} , respectively, have been defined, and that these spatial quantities are such that

$$\mathbf{G} \cdot \dot{\mathbf{A}} = \mathbf{\hat{G}} \cdot \dot{\mathbf{\hat{A}}} \quad , \quad (3.74)$$

where $(\dot{})$ denotes a suitable objective time rate.

We now define the array \mathbf{P} by

$$\mathbf{P} = (\mathbf{D}^P, \check{\mathbf{A}}, \dot{\gamma}) \quad (3.75)$$

and a conjugate array \mathbf{X} by

$$\mathbf{X} = (\text{dev } \bar{\sigma}, \Gamma, g) \quad (3.76)$$

If we use (3.75) and (3.76) in the reduced dissipation inequality (3.71) we get

$$\begin{aligned} g\dot{\gamma} + \mathbf{G} \cdot \dot{\mathbf{A}} + \text{dev } \bar{\sigma} \cdot \mathbf{D}^P &= g\dot{\gamma} + \Gamma \cdot \check{\mathbf{A}} + \text{dev } \bar{\sigma} \cdot \mathbf{D}^P \\ &= \mathbf{X} \cdot \mathbf{P} \\ &\geq 0 \end{aligned} \quad (3.77)$$

By comparison with the results of Chapter II we see that the dissipation inequality together with the notion of a closed convex region of admissible conjugate forces leads in a natural way to an evolution equation which can be postulated in three equivalent forms. Indeed, we assume now that the evolution equation takes the following form:

- (i) there exists a closed, convex region \mathbf{K} such that \mathbf{X} and \mathbf{P} are related by

$$\mathbf{P} \in N_{\mathbf{K}}(\mathbf{X}) \quad ; \quad (3.78)$$

(ii) equivalently,

$$\mathbf{X} \in \partial D(\mathbf{P}) \quad ; \quad (3.79)$$

(iii) equivalently,

$$\mathbf{X} \in \mathbb{H}(\mathbf{P}) \quad (3.80)$$

where \mathbb{H} is a maximal responsive map related to D and K as in Theorem 2.1.

Here $D = D_K$, the support function of K , and is assumed to possess the property (2.3), so that there exists a gauge f , polar to D , and a scalar $\lambda \geq 0$ such that

$$\mathbf{P} \in \lambda \partial f(\mathbf{X}) \quad . \quad (3.81)$$

The above evolution law is incomplete, since we must add information regarding the *skew* part of \mathbf{L}^P , in order to have complete information about the evolution of \mathbf{F}^P (see equations (3.36) and (3.37)). Here various options are available, the choice depending to some extent on the particular kind of elastic-plastic material being modelled. We describe two possibilities.

First, we recall the definition of the plastic spin tensor \mathbf{V}^P :

$$\mathbf{V}^P = \frac{1}{2} (\mathbf{L}^P - \mathbf{L}^{P^T}) \quad . \quad (3.82)$$

Then one appropriate form of the evolution law for the skew part of L^p is obtained simply by assuming that there is no plastic spin:

$$W^p = 0 \quad . \quad (3.83)$$

The physical appropriateness and implications of this assumption are discussed by Moran, Ortiz and Shih [55] and Needleman [59], for example. In this case

$$D^p = L^p = F^e \dot{F}^p F^{-1} \quad ,$$

whence one may find \dot{F}^p , given D^p , F and F^p .

A second option is to assume that F^p is *positive definite and symmetric*. Thus in equations (3.8) we choose $R^e = R$ so that

$$F^p = U^p$$

where U^p is the positive-definite symmetric part of the polar decomposition of F^p , and from (3.37) and (3.41) we have

$$D^p = \text{sym}(F^e \dot{U}^p U^{p-1} F^{e-1}) \quad (3.84)$$

so that once again it is possible to determine \dot{F}^p from a knowledge of D^p (when written out in component form relative to any basis, equation (3.84) represents a set of six equations for the six components of \dot{U}^p).

The material behaviour is thus completely described by the specification of ψ (whence we obtain the relations (3.66) and (3.70)), one of K , H or D (whence we construct the evolution law in one of the forms (3.78) - (3.81)), together with an assumption of the form (3.83) or (3.84).

An interesting alternative is discussed in [55], where the evolution law is constructed using quantities defined with respect to the *intermediate* configuration, and where in particular a convex yield function is defined with respect to such quantities. In the case of metals, for which elastic strains are small, there would presumably be little difference between evolution laws which differ only in their use of intermediate rather than spatial quantities as primary variables. We have chosen to make use of spatial variables, but should there be compelling physical evidence for the suitability of intermediate variables, the theory presented here can be modified with little effort, by replacing conjugate pairs of spatial variables with corresponding conjugate intermediate variables.

In problems involving elastic-plastic materials it is found that the evolution equation is most commonly used in the form in which it is expressed as a normality condition (equations (3.78) or (3.81)). It has been demonstrated [50,51,52], however, that there are considerable advantages to using an evolution equation in the form (3.79) involving the dissipation function, particularly when numerical solutions to the corresponding initial-boundary value problems are sought. The properties of the dissipation function,

particularly its convexity and positive homogeneity, allow one to construct rational approximations which in turn place in perspective the well-known schemes for integration of constitutive equations (see [51] for a review).

In order to recover the plastic constitutive equations in their conventional classical form it is necessary to add a further axiom, namely, the *consistency condition*. This states that for $\mathbf{X} \in \text{bd } K$,

$$\lambda \dot{f} = 0 \quad (3.85)$$

where \dot{f} is the change in the canonical yield function $f(\mathbf{X})$ associated with the change in \mathbf{X} accompanying \mathbf{P} . Thus for a plastic state we have

$$\lambda > 0, \quad \dot{f} = 0 \quad (3.86)$$

We can recover the normality rule (3.73) in its classical form by part (d) of Lemma 2.2, that is equation (3.81), or if f is smooth we use (2.21) to give

$$\mathbf{P} = \lambda \nabla f(\mathbf{X}) \quad (3.87)$$

3.5 AN EXAMPLE OF A FREE ENERGY FUNCTION

We consider an example of an elastic-plastic material which exhibits both kinematic as well as isotropic hardening. Kinematic

hardening is accounted for by the introduction of an internal variable \mathbf{A} which is assumed symmetric, and isotropic hardening is represented by a scalar γ , which could be the total plastic work or the effective plastic strain, for example.

We consider a free energy function in the separable form

$$\psi(\mathbf{F}^e, \gamma, \mathbf{A}) = \psi_0(\mathbf{F}^e) + \psi_1(\mathbf{A}) + \psi_2(\gamma) \quad , \quad (3.88)$$

and proceed to make suitable choices for ψ_0 , ψ_1 and ψ_2 . In seeking an appropriate function ψ_0 we are guided by the notion that when the material is elastic (so that $\mathbf{F}^e = \mathbf{F}$) the corresponding boundary value problem should be well-posed, in that at least one solution exists in a suitable Sobolev space. While there is as yet no existence theory for elastic-plastic problems with finite strains, it is reasonable to assume that such a theory, based on a free energy of the form (3.88), would require the function ψ_0 to have at least the properties which the elastic theory demands, for existence of solutions. Realistic sufficient conditions for this to be the case in elasticity were first given by Ball [4] (see Ciarlet [18] for an excellent account of this theory). In the present context these conditions amount to the following: it is required that

- (a) ψ_0 be polyconvex, that is, ψ_0 be expressible as a convex function of \mathbf{F}^e , $\text{cof } \mathbf{F}^e$ and $\text{cof } \mathbf{F}^e$, where $\text{cof } \mathbf{A}$ denotes the cofactor of \mathbf{A} : $\text{cof } \mathbf{A} = (\det \mathbf{A})\mathbf{A}^{-T}$ for invertible \mathbf{A} ;

(b) $\psi_0(\mathbf{F}^e) \rightarrow +\infty$ as $\det \mathbf{F}^e \rightarrow 0^+$;

(c) ψ_0 be coercive, in the sense that there exist constants α, β, p, q, r such that

$$\psi_0(\mathbf{F}^e) \geq \alpha(|\mathbf{F}^e|^p + |\operatorname{cof} \mathbf{F}^e|^q + (\det \mathbf{F}^e)^r) + \beta$$

for all invertible \mathbf{F}^e , where $\alpha > 0$, $p \geq 2$, $q \geq \frac{p}{p-1}$, $r > 1$.

For example, we may adopt a free energy function proposed by Ciarlet and Geymonat [18] (see also [17]) which possesses all of the above properties and which has the further merit that it is easily adjustable to experimental results; this function is of the form

$$\psi_0(\mathbf{F}^e) = a|\mathbf{F}^e|^2 + b|\operatorname{cof} \mathbf{F}^e|^2 + \ell(\det \mathbf{F}^e) + e \quad (3.89)$$

where $a > 0$, $b > 0$, $e \in \mathbb{R}$ and

$$\ell(\delta) = c\delta^2 - d \log \delta, \quad ,$$

with $c > 0$, $d > 0$. If we use the expressions

$$|\mathbf{F}^e|^2 = \operatorname{tr} \mathbf{F}^{eT} \mathbf{F}^e = \operatorname{tr} \mathbf{C}^e$$

and

$$|\operatorname{cof} \mathbf{F}^e|^2 = \frac{1}{2} (\operatorname{tr} \mathbf{F}^{eT} \mathbf{F}^e)^2 - \frac{1}{2} \operatorname{tr} (\mathbf{F}^{eT} \mathbf{F}^e)^2 = \frac{1}{2} (\operatorname{tr} \mathbf{C}^e)^2 - \frac{1}{2} \operatorname{tr} (\mathbf{C}^e)^2$$

we can express (3.89) as a function of \mathbf{C}^e to obtain

$$\begin{aligned} \psi_0(\mathbf{C}^e) = & a \operatorname{tr} \mathbf{C}^e + \frac{b}{2} (\operatorname{tr} \mathbf{C}^e)^2 - \frac{b}{2} \operatorname{tr} (\mathbf{C}^e)^2 + c \det \mathbf{C}^e \\ & - \frac{d}{2} \log (\det \mathbf{C}^e) + e \quad . \end{aligned} \quad (3.90)$$

Next, we consider the term ψ_1 involving the internal variable \mathbf{A} . The conjugate force \mathbf{G} is, from (3.66),

$$\mathbf{G} = - \rho_0 \frac{\partial \psi_1}{\partial \mathbf{A}} \quad .$$

We treat \mathbf{G} as a stress-like referential quantity and define its spatial counterpart $\mathbf{\Gamma}$ by

$$\mathbf{\Gamma} = \mathbf{F} \mathbf{G} \mathbf{F}^T$$

so that if the spatial counterpart $\mathbf{\Lambda}$ of the internal variable \mathbf{A} is defined by

$$\mathbf{\Lambda} = \mathbf{F}^{-T} \mathbf{A} \mathbf{F}^{-1} \quad , \quad (3.91)$$

we find that the contribution to the dissipation of this internal variable is (after expressing $\mathbf{\Lambda} = \mathbf{F}^T \mathbf{A} \mathbf{F}$ from (3.91) and finding the time derivative of $\mathbf{\Lambda}$ as $\dot{\mathbf{\Lambda}} = \mathbf{F}^T \check{\mathbf{A}} \mathbf{F}$)

$$\mathbf{G} \cdot \dot{\mathbf{\Lambda}} = \mathbf{\Gamma} \cdot \check{\mathbf{A}}$$

where $(\dot{})$ represents the objective time rate defined by

$$\check{\mathbf{A}} = \dot{\mathbf{A}} + \mathbf{L}^T \mathbf{A} + \mathbf{A} \mathbf{L} \quad .$$

Thus the internal variable $\check{\mathbf{A}}$ which appears as a component of \mathbf{P} in (3.75) is defined in a natural way, as an appropriate conjugate quantity.

For consistency with the form which the flow law will take it is necessary also to assume that $\check{\mathbf{A}}$, like \mathbf{D}^P , is trace-free. Then the dissipation inequality (3.71) can be written in the form

$$g\dot{\gamma} + \text{dev } \Gamma \cdot \check{\mathbf{A}} + \text{dev } \bar{\sigma} \cdot \mathbf{D}^P \geq 0 \quad , \quad (3.92)$$

with the result that the ordered triples \mathbf{P} and \mathbf{X} take the form

$$\mathbf{P} = (\mathbf{D}^P, \check{\mathbf{A}}, \dot{\gamma}) \quad , \quad \mathbf{X} = (\text{dev } \bar{\sigma}, \text{dev } \Gamma, g) \quad . \quad (3.93)$$

In order to accommodate isotropic hardening we let the internal variable γ represent the accumulated plastic work or some scalar measure of accumulated plastic strain, and consider an expression for ψ_2 of the form

$$\psi_2(\gamma) = \frac{1}{2} c_1 \gamma^2 + (c_2 - c_0)(\gamma + \beta^{-1} \exp(-\beta\gamma)) \quad (3.94)$$

where c_0 , c_1 , c_2 and β are material constants with $c_2 > c_0$ (a similar example is treated in [37]), and of course the force g conjugate to γ

is given by

$$g = - \rho_0 \frac{\partial \psi_2}{\partial \gamma} \quad (3.95)$$

Finally, we need to specify one of \mathbb{H} , D or K . We choose for simplicity a von Mises yield condition, suitably altered to accommodate kinematic and isotropic hardening, so that the yield condition is of the form

$$|\text{dev}(\bar{\sigma} + \Gamma)| \leq c_0 + c_1 \gamma + (c_2 - c_0)(1 - \exp(-\beta \gamma)) \quad (3.96)$$

where $c_0^2 = \frac{2}{3} \bar{\sigma}_0^2$, $\bar{\sigma}_0$ being the yield stress in uniaxial tension. The canonical yield function is thus

$$f(\mathbf{X}) = \frac{|\text{dev}(\bar{\sigma} + \Gamma)| + g}{c_0} \quad (3.97)$$

It is established that f is a gauge: it clearly satisfies the conditions $f(\mathbf{X}) \geq 0$, $f(0) = 0$, and it is positively homogeneous, so it remains to check lower semicontinuity and convexity. f is in fact continuous, hence lower semicontinuous. Convexity follows from the linearity of dev and repeated use of the triangle inequality using the norm $|\cdot|$ on E .

The flow law (3.87) now gives

$$\mathbf{P} \equiv \begin{bmatrix} \mathbf{D}^P \\ \check{\mathbf{A}} \\ \dot{\gamma} \end{bmatrix} = \lambda \begin{bmatrix} \mathbf{N} \\ \mathbf{N} \\ 1 \end{bmatrix} \quad (3.98)$$

where the constant c_0 has been absorbed into λ , and

$$N = \frac{\text{dev}(\bar{\sigma} + \Gamma)}{|\text{dev}(\bar{\sigma} + \Gamma)|}.$$

Thus we find that

$$\dot{\gamma} = \lambda \quad \text{and} \quad \mathbf{D}^P = \check{\mathbf{A}}. \quad (3.99)$$

In view of (3.98) and (3.99) we may restrict attention to the subspace consisting of those vectors \mathbf{P} of the form $\mathbf{P} = (\mathbf{Z}, \mathbf{Z}, a)$ with $\text{tr } \mathbf{Z} = 0$, and for vectors in this subspace the dissipation function D is

$$\begin{aligned} D(\mathbf{P}) &= \sup_{f(\mathbf{X}) \leq 1} \mathbf{X} \cdot \mathbf{P} \\ &= \sup_{f(\mathbf{X}) = 1} \mathbf{X} \cdot \mathbf{P} \\ &= \sup_{f(\mathbf{X}) = 1} \{\text{dev}(\bar{\sigma} + \Gamma) \cdot \mathbf{D}^P + g\dot{\gamma}\} \\ &= \sup_{f(\mathbf{X}) = 1} \{g(\dot{\gamma} - |\mathbf{D}^P| \cos \theta) + c_0 |\mathbf{D}^P| \cos \theta\} \quad (3.100) \end{aligned}$$

where

$$\cos \theta = \frac{(\text{dev}(\bar{\sigma} + \Gamma)) \cdot \mathbf{D}^P}{|\text{dev}(\bar{\sigma} + \Gamma)| |\mathbf{D}^P|}$$

Three situations are possible: $\dot{\gamma}$ is either less than, or equal to, or greater than, $|\mathbf{D}^P|$. For each of these situations we evaluate the supremum on the right hand side of (3.98) to obtain

$$\begin{aligned} \text{(I)} \quad \dot{\gamma} < |\mathbf{D}^P| &: D(\mathbf{P}) \rightarrow \infty \\ \text{(II)} \quad \dot{\gamma} &= |\mathbf{D}^P| : D(\mathbf{P}) = \dot{\gamma} \\ \text{(III)} \quad \dot{\gamma} > |\mathbf{D}^P| &: D(\mathbf{P}) = \dot{\gamma} \end{aligned} \quad (3.101)$$

The situation is summarised pictorially in Figure 3.2, which shows for a one-dimensional situation the canonical yield function and the dissipation function. We note that while the theory gives rise to the three possibilities summarised in (3.101), in practice it is only (II) which will be possible: the region $g > 0$ in the set K is not accessible since we have at all times $\gamma \geq 0 \Leftrightarrow g \leq 0$, so that case (III) will not arise. It is clear from the normality law and the shape of K that case (I) will not arise either.

3.6 NON-SMOOTH YIELD SURFACES

Although the theory of plasticity presented in this thesis is broad enough to cover convex non-smooth yield surfaces, in subsequent Chapters we will need an explicit equation for the normality

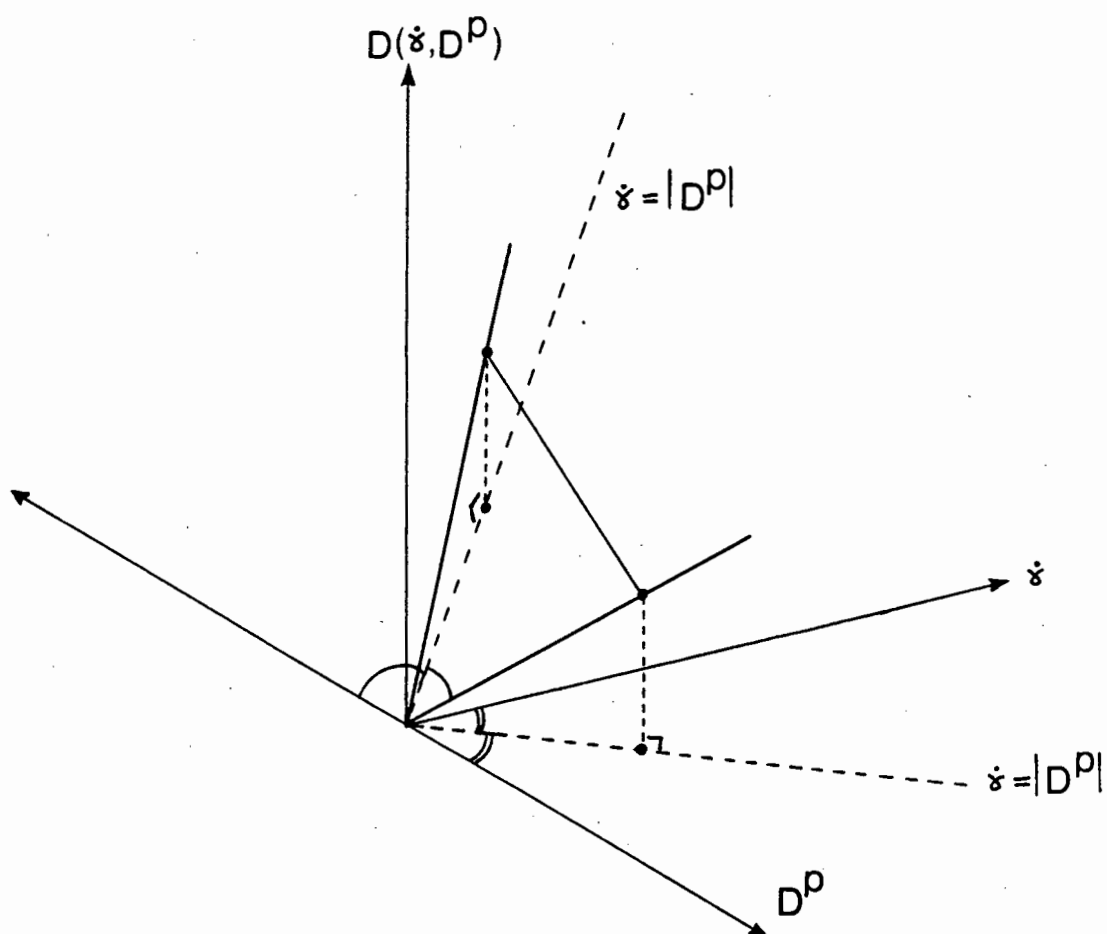
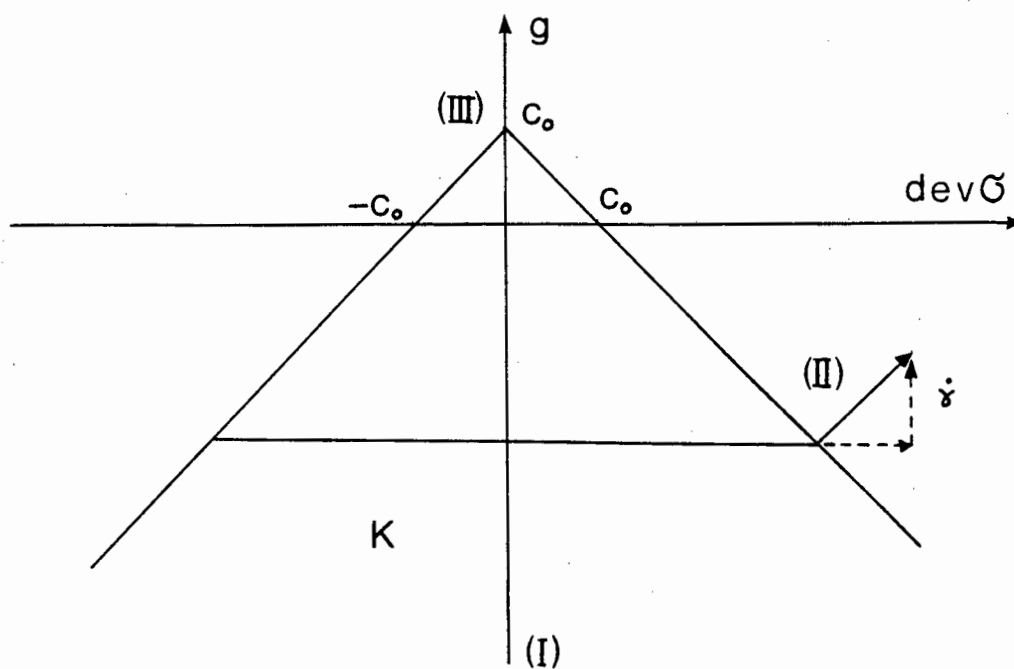


Figure 3.2 The region K and the associated dissipation function D for the case in which kinematic hardening is suppressed and $\bar{\sigma}$ and D^p are taken to be one-dimensional.

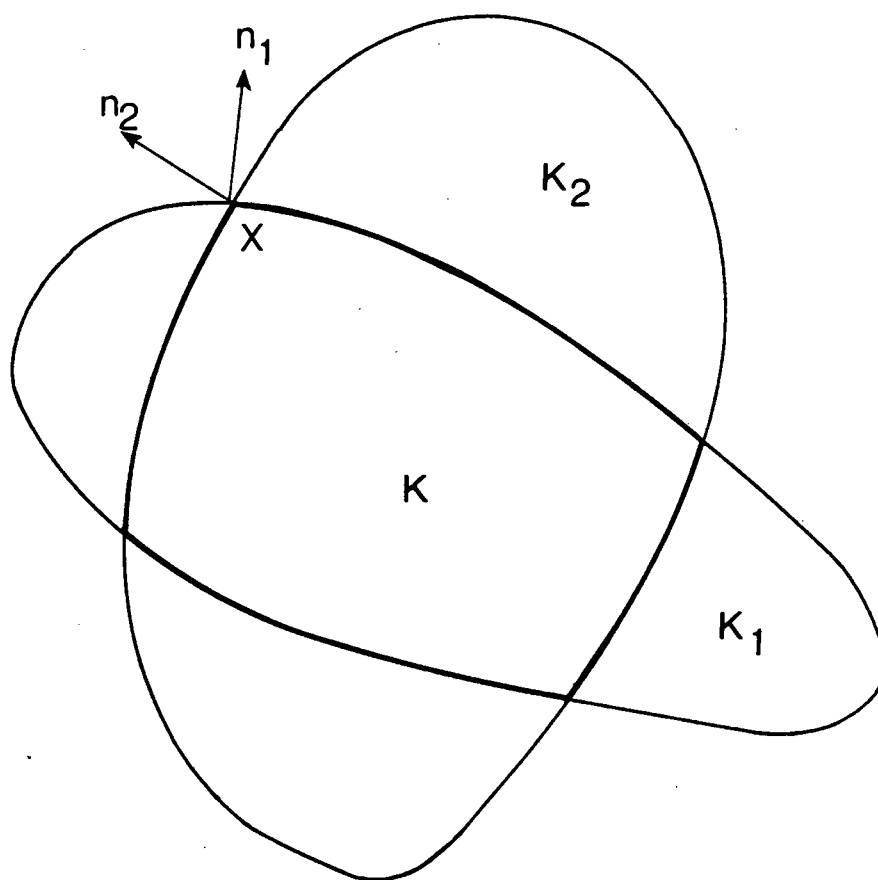


Figure 3.3 The set $K = K_1 \cap K_2$ where the solid lines show $\text{bd } K$ and n_1, n_2 denote the limiting normals at X .

condition similar to the form (3.87), at points of a yield surface which are not smooth. This is because the array P is discontinuous across an acceleration wave front, and we will need to find the jump in P across a singular surface. The expression (3.81) of P at a non-smooth point on the yield surface is inadequate for this purpose.

The von Mises yield criterion gives a smooth surface, whereas another commonly used yield criterion, the Tresca yield criterion, gives a non-smooth yield surface. In order to arrive at a solution to this problem we will incorporate Sewell's theory [70,71] for non-smooth yield surfaces into the theory presented earlier.

Sewell's theory is limited to smooth surfaces in 3-dimensional space whose intersections are lines or, in the special case of plane stress, smooth curves whose intersections are points. In the case of yield surfaces in 3-dimensional space which have singularities in the form of lines rather than points, the cone of normals at corners is finite-dimensional. The theory treats only the case of isotropic hardening; hence, we also work within this restriction.

In this section we briefly present Sewell's theory and explain how it can be incorporated into our theory. The section concludes with expressions for stress rate which will be of use in the next Chapter.

We assume that the non-smooth region of all admissible stresses is an intersection of m separate regions, each with a smooth

boundary. Hence the *piecewise smooth* yield surface is the boundary of the intersection of these regions, that is,

$$K = \{\mathbf{X} : f_\alpha(\mathbf{X}) < 1 \quad \text{for all } \alpha\}$$

where f_α is the canonical yield function associated with K_α and

$$K_\alpha = \{\mathbf{X} : f_\alpha(\mathbf{X}) < 1\} \quad ,$$

so that

$$K = \bigcap_{\alpha} K_\alpha \quad .$$

The region \bar{K} which is

$$\bar{K} \equiv K \cup \text{bd } K$$

is closed, and is convex if each K_α is convex. The region K is illustrated in Figure 3.3 with the *normals* $\mathbf{n}_\alpha(\mathbf{X})$ for $\alpha = 1, 2$. The unique normals $\mathbf{n}_\alpha(\mathbf{X})$ corresponding to the convex set K , at a non-smooth point \mathbf{X} lie at the boundary $\text{bd } N$ of the normal cone $N_K(\mathbf{X})$, with each \mathbf{n}_α corresponding to K_α (with the restriction to surfaces whose intersections are lines or smooth curves whose intersections are points).

If we consider only isotropic hardening and introduce scalars γ_α ($\alpha = 1, \dots, m$) associated with the regions K_α , (3.88) becomes

$$\psi(\mathbf{F}^e, \gamma_\alpha) = \psi_0(\mathbf{F}^e) + \psi_2(\gamma_\alpha)$$

which can be transformed to

$$\psi(\mathbf{C}^e, \gamma_\alpha) = \psi_0(\mathbf{C}^e) + \psi_2(\gamma_\alpha) \quad (3.102)$$

by using material-frame indifference.

The scalar parameters γ_α are unknown in advance, but for each a we have the conditions

$$\dot{\sigma} \cdot \mathbf{M}^\alpha - h_{\alpha\beta} \dot{\gamma}_\beta \leq 0 \quad , \quad \dot{\gamma}_\alpha \geq 0 \quad , \quad (3.103)$$

$$\dot{\gamma}_\alpha (\dot{\sigma} \cdot \mathbf{M}^\alpha - h_{\alpha\beta} \dot{\gamma}_\beta) = 0 \quad (\text{no sum on } a) \quad , \quad (3.104)$$

where $h_{\alpha\beta}$ are the elements of a given $m \times m$ symmetric *hardening matrix*, assumed positive definite, and $\mathbf{M}^\alpha = \partial f^\alpha / \partial \bar{\sigma}$. Here and henceforth we sum on repeated sub- or superscripts a, β unless otherwise stated. The inequalities (3.103) imply that, for each a , strict choice in one inequality implies an equation in the other, and in the orthogonality (normality) condition (3.104) each term in the a sum must be zero. Summation on β is implied in these equations, however. In the case of a smooth yield surface inequalities (3.103) and equation (3.104) reduce to

$$\dot{\sigma} \cdot \mathbf{M} - h \dot{\gamma} \leq 0 \quad , \quad \dot{\gamma} \geq 0 \quad ;$$

$$\dot{\gamma} (\dot{\sigma} \cdot \mathbf{M} - h \dot{\gamma}) = 0$$

for $m = a = \beta = 1$ where $\dot{\gamma} = \dot{\gamma}_1$, $\mathbf{M} = \mathbf{M}^1$ and $h = h_{11}$.

The strict inequalities $\dot{\gamma}_\alpha > 0$ may apply for only a subset of the set of m α -values. This is called the *active subset*, and the remaining subset for which each $\dot{\gamma}_\alpha = 0$ the *passive subset*. The occurrence of one or more strict inequalities, that is, the existence of an active subset, implies the existence of a *plastic loading state* within the material body. *Fully active loading* occurs when all $\dot{\gamma}_\alpha > 0$. For each active surface K_α , (3.104) gives

$$\dot{\gamma}_\alpha = h_{\alpha\beta}^{-1} \mathbf{M}^\beta \cdot \dot{\boldsymbol{\sigma}} > 0 \quad . \quad (3.105)$$

The conjugate force g_α (3.95) corresponding to γ_α is defined by

$$g_\alpha = - \rho_0 \frac{\partial \psi_2}{\partial \gamma_\alpha} \quad . \quad (3.106)$$

We define a set of yield functions of the form

$$f_\alpha(\mathbf{X}) = f_{0\alpha}(\bar{\boldsymbol{\sigma}}) + g_\alpha \quad \text{for all } \alpha , \quad (3.107)$$

where \mathbf{X} is the reduced form of (3.76) by ignoring kinematic hardening, hence setting $\boldsymbol{\Gamma} = \mathbf{0}$. The expression (3.77) for \mathbf{P} becomes

$$\mathbf{P} = (\mathbf{D}^P, \dot{\gamma}_\alpha) \quad (3.108)$$

or, in view of (3.87),

$$\mathbf{D}^P = \sum_{a=1}^m \lambda_a \mathbf{H}^a \quad (3.109)$$

for m separate surfaces and active values of $\dot{\gamma}_a$, where

$$\dot{\gamma}_a = \lambda_a \quad \text{for each } a \quad (3.110)$$

which is found from (3.87) similar to (3.98), and

$$\mathbf{H}^a = \frac{\partial f_{0a}}{\partial \bar{\sigma}} \quad (3.111)$$

In order to obtain the rate form of the equations for plastic loading we begin with (3.70) and obtain the time derivative of $\bar{\sigma}$, that is,

$$\dot{\bar{\sigma}} = 2\rho_0 \left[\dot{\mathbf{F}}^e \frac{\partial \psi}{\partial \mathbf{C}}^e \mathbf{F}^{eT} + \mathbf{F}^e \frac{\partial \psi}{\partial \mathbf{C}}^e \dot{\mathbf{F}}^{eT} + \mathbf{F}^e \frac{\partial^2 \psi}{\partial \mathbf{C}^e \partial \mathbf{C}}^e \dot{\mathbf{C}}^e \mathbf{F}^{eT} \right] \quad (3.112)$$

It is possible to express (3.112) in terms of \mathbf{D}^P ; to do this we need to write $\dot{\mathbf{F}}^e$ and $\dot{\mathbf{C}}^e$ in terms of \mathbf{D}^P . We will do it by incorporating, beforehand, the constitutive assumption (3.83). From the additive split $\mathbf{W} = \mathbf{W}^e + \mathbf{W}^P$ of the spin tensor into its elastic

and plastic parts \mathbf{W}^e and \mathbf{W}^p and (3.83) we obtain the results

$$\mathbf{W} = \mathbf{W}^e$$

and

$$\mathbf{L}^p = \mathbf{L}^{pT} = \mathbf{D}^p \quad . \quad (3.113)$$

If we use the equations (3.35), (3.39) in (3.43), and (3.44) we obtain the results

$$\begin{aligned} \dot{\mathbf{F}}^e &= (\mathbf{L} - \mathbf{D}^p) \mathbf{F}^e \quad , \\ \dot{\mathbf{C}}^e &= 2\mathbf{F}^{eT} (\mathbf{D} - \mathbf{D}^p) \mathbf{F}^e \quad . \end{aligned} \quad (3.114)$$

If we use the equations (3.114) in (3.112) we obtain

$$\dot{\bar{\sigma}} = (\mathbf{L} - \mathbf{D}^p) \bar{\sigma} + \bar{\sigma} (\mathbf{L}^T - \mathbf{D}^p) + 2\mathbf{F}^e \mathbf{B} (\mathbf{F}^{eT} (\mathbf{D} - \mathbf{D}^p) \mathbf{F}^e) \mathbf{F}^{eT} \quad (3.115)$$

where \mathbf{B} is a fourth order elasticity tensor defined by

$$\mathbf{B} = 2\rho_0 \frac{\partial^2 \psi}{\partial \mathbf{C}^e \partial \mathbf{C}^e} \quad . \quad (3.116)$$

In view of (3.113), (3.115) becomes

$$\dot{\bar{\sigma}} = \mathbf{D}^e \bar{\sigma} + \bar{\sigma} \mathbf{D}^e - \bar{\sigma} \mathbf{W} + \xi(\mathbf{D}^e) + \mathbf{W} \bar{\sigma} \quad (3.117)$$

where the components of the fourth order tensor ξ are defined by

$$\xi_{ijmn} = 2B_{\alpha\beta\lambda\mu} F_{i\alpha}^e F_{j\beta}^e F_{m\lambda}^e F_{n\mu}^e \quad . \quad (3.118)$$

If we define the co-rotational (or Jaumann) *rate of stress* $\overset{\circ}{\bar{\sigma}}$ by

$$\overset{\circ}{\bar{\sigma}} = \dot{\bar{\sigma}} - \mathbf{W}\bar{\sigma} + \bar{\sigma}\mathbf{W} \quad , \quad (3.119)$$

then (3.119) becomes

$$\overset{\circ}{\bar{\sigma}} = \mathbf{D}^e \bar{\sigma} + \bar{\sigma} \mathbf{D}^e + \xi(\mathbf{D}^e) \quad (3.120)$$

with components given by

$$\overset{\circ}{\bar{\sigma}}_{ij} = \xi_{ijkl} D_{kl}^e + D_{ip}^e \bar{\sigma}_{pj} + \bar{\sigma}_{ip} D_{pj}^e = \mathcal{L}_{ijkl} D_{kl}^e \quad (3.121)$$

where the fourth order tensor \mathcal{L} is defined in component form by

$$\mathcal{L}_{ijkl} = \xi_{ijkl} + \delta_{ik} \bar{\sigma}_{lj} + \bar{\sigma}_{ik} \delta_{jl} \quad . \quad (3.122)$$

If we substitute (3.39), (3.121) becomes

$$\overset{\circ}{\bar{\sigma}} = \mathcal{L}(\mathbf{D}) - \mathcal{L}(\mathbf{D}^p) \quad . \quad (3.123)$$

We can obtain the following useful results:

$$(i) \quad tr \dot{\bar{\sigma}} = tr \check{\bar{\sigma}} \quad , \quad (3.124)$$

$$(ii) \quad \bar{\sigma} \cdot \dot{\bar{\sigma}} = \bar{\sigma} \cdot \check{\bar{\sigma}} \quad , \quad (3.125)$$

$$(iii) \quad \bar{\sigma}^{-1} \cdot \dot{\bar{\sigma}} = \bar{\sigma}^{-1} \cdot \check{\bar{\sigma}} \quad . \quad (3.126)$$

Using (3.119) and (3.126) we obtain another useful result

$$\mathbf{H}^{\alpha} \cdot \dot{\bar{\sigma}} = \mathbf{H}^{\alpha} \cdot \check{\bar{\sigma}} \quad , \quad (3.127)$$

and from (3.107) and (3.127) we recover the consistency condition (3.86) as

$$\mathbf{H}^{\alpha} \cdot \check{\bar{\sigma}} - \rho_0 \psi_{2, \alpha\beta} \lambda_{\beta} = 0 \quad , \quad \text{for each } \alpha \quad . \quad (3.128)$$

Hence, (3.130) is another version of equation (3.104), and we make the identification $\rho_0 \psi_{2, \alpha\beta} = h_{\alpha\beta}$.

If we substitute (3.109) in (3.123) we obtain

$$\check{\bar{\sigma}} = \mathcal{L}(\mathbf{D}) - \sum_{\beta=1}^m \lambda_{\beta} \mathcal{L}(\mathbf{H}^{\beta}) \quad . \quad (3.129)$$

Substitution in (3.128) yields

$$h_{\alpha\beta}\lambda_\beta = \mathbf{M}^\alpha \cdot (\mathcal{L}(\mathbf{D}) - \lambda_\beta \mathcal{L}(\mathbf{M}^\beta)) \quad ,$$

or

$$\lambda_\beta = G_{\alpha\beta}^{-1} \mathbf{M}^\alpha \cdot (\mathcal{L}(\mathbf{D})) \quad (3.130)$$

where the positive-definite symmetric matrix $G_{\alpha\beta}$ is defined by

$$G_{\alpha\beta} = h_{\alpha\beta} + \mathbf{M}^\alpha \cdot \mathcal{L}(\mathbf{M}^\beta) \quad . \quad (3.131)$$

For the case of two independent yield functions we choose a *hardening matrix* $h_{\alpha\beta} = \rho_0 \psi_{2,\alpha\beta}$ of the form

$$h_{\alpha\beta} = h \begin{bmatrix} 1 & \phi \\ \phi & 1 \end{bmatrix} \quad (m = 2 \text{ here}) \quad (3.132)$$

where

$$h > 0 \quad \text{and} \quad -1 < \phi < 1 \quad .$$

Finally by using (3.130) in (3.129) we obtain

$$\bar{\sigma} = \mathcal{L}(\mathbf{D}) - G_{\alpha\beta}^{-1} \{ \mathbf{M}^\alpha \cdot \mathcal{L}(\mathbf{D}) \} \mathcal{L}(\mathbf{M}^\beta) \equiv \mathcal{A}(\mathbf{D}) \quad (3.133)$$

where \mathcal{A} is the fourth order tensor given by

$$\mathcal{A} = \mathcal{L} - G_{\alpha\beta}^{-1} \mathcal{L}(\mathbf{M}^\alpha) \otimes \mathcal{L}(\mathbf{M}^\beta) \quad . \quad (3.134)$$

For a smooth yield surface we have $\alpha = \beta = 1$, or $\gamma^\alpha = \gamma^\beta = \gamma$ and

$$\mathcal{A} = \mathcal{L} - G_{11}^{-1} \mathcal{L}(\mathbf{M}) \otimes \mathcal{L}(\mathbf{M}) \quad , \quad (3.135)$$

with

$$G_{11} = h + \mathbf{M} \cdot \mathcal{L}(\mathbf{M}) \quad . \quad (3.136)$$

The parameter ϕ represents the amount of "coupled hardening" which is being assumed. Physical information on the value to be assigned to ϕ is limited. Some well-known cases are $\phi = 0$, which implies no coupling and is called *independent hardening* by Koiter, and $\phi = 1$, which is the *isotropic hardening* of Taylor (not to be confused with the general notion of "isotropic hardening"). (See [70] for further information on the hardening matrix and relations between its parameters.)

The abovementioned results will later on be used extensively in the discussion of the propagation of acceleration waves in plastic media whose states of stress lie on yield surface vertices or at smooth points. Hence, the relevant examples of the use of particular yield surfaces of this Section will be found in Chapter 5.

CHAPTER IV

SINGULAR SURFACES AND ACCELERATION WAVES

In this Chapter we review the basic concepts of the theory of singular surfaces paying particular attention to the propagation of acceleration waves, and we present some standard applications in relation to acceleration waves in elastic solid media. The material reviewed in this Chapter is covered more extensively in, for example, Truesdell and Toupin [70], Chen [16], Eringen and Suhubi [25], and McCarthy [53].

We define a singular surface and record the geometric and kinematic compatibility conditions to be satisfied across a propagating singular surface. We then give the definitions of principal, longitudinal and transverse waves, and conclude the Chapter with derivations of the equations of motion of acceleration waves in the context of finite elasticity, in order to provide a basis for comparison with the results for waves in plastic media which will be presented in the following Chapter.

Singular Surfaces

We denote by $S(t)$ a one-parameter family of surfaces, that is, a moving surface in Euclidean point space E . In reference

coordinates the position of the surface at time t is described by the equation

$$\Sigma(\mathbf{X}, t) = 0 \quad ; \quad (4.1)$$

the surface is assumed to be at least continuously differentiable, and of arbitrary shape. $S(t)$ and its image $s(t)$ in the current configuration are illustrated in Figure 4.1. The surface has an alternative representation by the use of a pair of curvilinear surface coordinates Y^α ($\alpha = 1, 2$), in which case the position vector \mathbf{X} of a point on the surface is

$$\mathbf{X} = \mathbf{X}(Y^\alpha, t) \quad , \quad (4.2)$$

and the corresponding tangent basis at \mathbf{X} is given by

$$\mathbf{H}_\alpha = \frac{\partial \mathbf{X}}{\partial Y^\alpha} \quad . \quad (4.3)$$

The unit normal \mathbf{N} to the surface is

$$\mathbf{N} = \frac{\text{Grad } \Sigma}{|\text{Grad } \Sigma|} \quad , \quad (4.4)$$

which has the alternative expression

$$\mathbf{N} = \frac{\mathbf{H}_1 \times \mathbf{H}_2}{|\mathbf{H}_1 \times \mathbf{H}_2|} \quad . \quad (4.5)$$

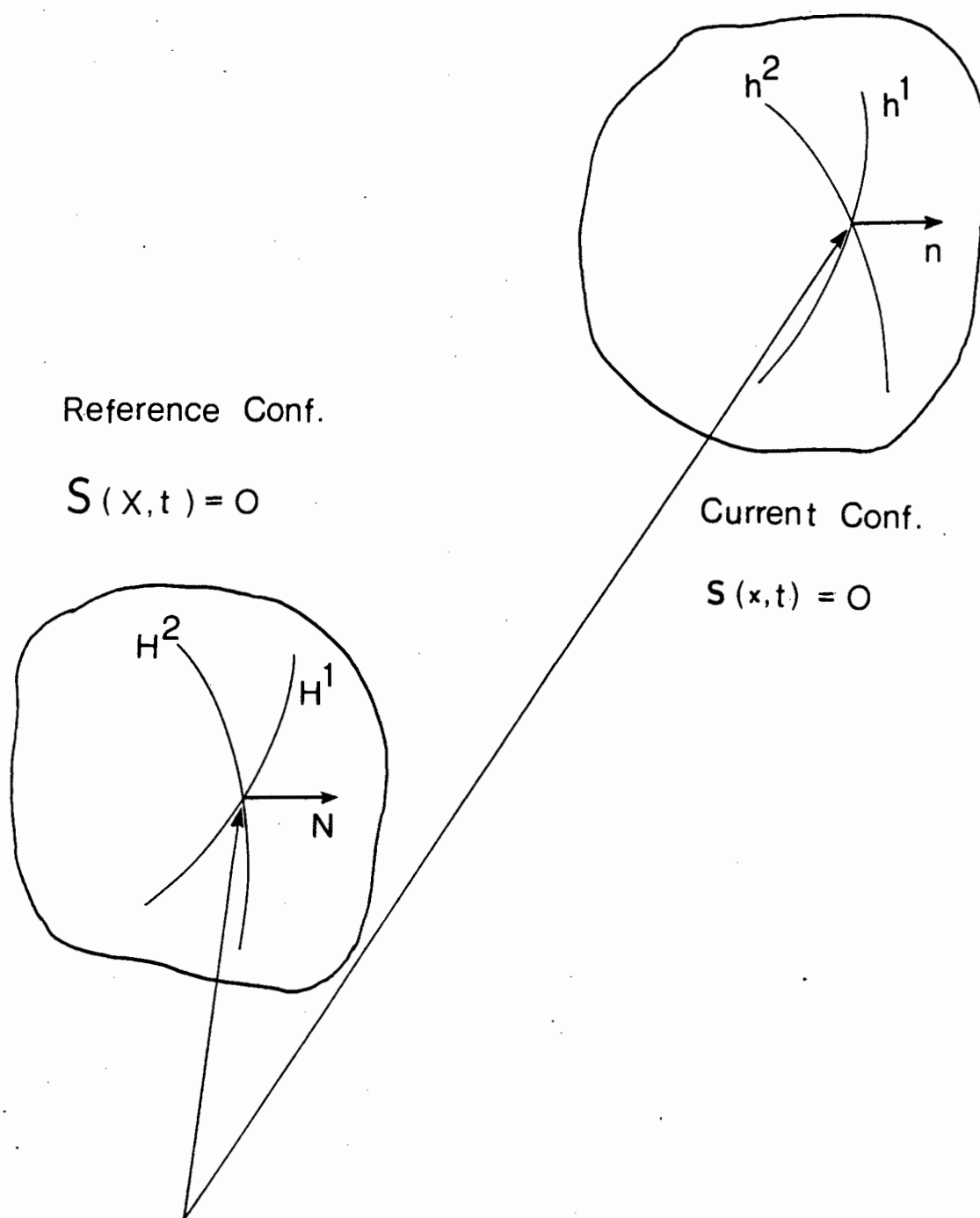


Figure 4.1 Surface S in reference configuration and its image s in current configuration.

The velocity \mathbf{u} of a point on the surface is

$$\mathbf{u} = \frac{\partial}{\partial t} (\mathbf{X}(Y^\alpha, t)) \quad (4.6)$$

and the normal velocity of the surface, or the *speed of displacement*, is defined by

$$U \equiv \mathbf{u} \cdot \mathbf{N} = N_i \frac{\partial u_i}{\partial t} \quad (4.7)$$

The speed of displacement is independent of the specific choice of the surface coordinates Y^α , whereas the velocity \mathbf{u} generally depends on a particular choice. If we choose the surface coordinates in such a way that $\mathbf{u} = U\mathbf{N}$, then the trajectories of surface points become equivalent to the trajectories of the one-parameter family of surfaces $S(t)$. Hence, if U is independent of the surface coordinates, then the configurations of the moving surface constitute a family of *parallel surfaces*. U is a measure of the speed with which the surface $S(t)$ traverses the material body.

The image $\mathbf{s}(t)$ of the surface $S(t)$ in the current configuration is defined by

$$\sigma(\mathbf{x}, t) \equiv \Sigma(\mathbf{X}, t) \quad (4.8)$$

where the relation between \mathbf{x} and \mathbf{X} is given by the motion (3.1).

If we use equation (4.2) we obtain

$$\mathbf{x} = \mathbf{x}(\mathbf{X}(Y^\alpha, t), t) \equiv \bar{\mathbf{x}}(Y^\alpha, t) \quad , \quad (4.9)$$

which provides an alternative expression to (4.8).

The tangent basis vectors \mathbf{h}_α on $s(t)$ are given by

$$\mathbf{h}_\alpha = \frac{\partial \bar{\mathbf{x}}}{\partial Y^\alpha} \quad (4.10)$$

and the unit normal \mathbf{n} to $s(t)$ is

$$\mathbf{n} = \frac{\text{grad } \sigma}{|\text{grad } \sigma|} = \frac{\mathbf{h}_1 \times \mathbf{h}_2}{|\mathbf{h}_1 \times \mathbf{h}_2|} \quad , \quad (4.11)$$

where grad denotes the gradient operator in the current configuration.

The unit normal vector \mathbf{n} can be expressed in terms of \mathbf{N} by using the deformation gradient tensor \mathbf{F} . Hence the relation between the unit normal vectors to the surfaces in the reference and current configurations is given by

$$\mathbf{N} = \frac{\mathbf{F}^T \mathbf{n}}{|\mathbf{F}^T \mathbf{n}|} \quad \Leftrightarrow \quad \mathbf{n} = \frac{\mathbf{F}^{-T} \mathbf{N}}{|\mathbf{F}^{-T} \mathbf{N}|} \quad . \quad (4.12)$$

Equations (4.12) are obtained from the relation between the differential vector $d\mathbf{X}$ in the reference configuration and its image $d\mathbf{x}$ in the current configuration (that is, $d\mathbf{x} = \mathbf{F}d\mathbf{X}$) and from the relation between differential area elements in the reference and current configurations.

Although the equations (4.1) and (4.9) define the same surface, they differ from each other in the sense that (4.9) gives the geometry of the surface at time t whereas (4.1) is the locus of initial positions of the points \mathbf{X} that lie upon the surface $\sigma(\mathbf{x}, t)$ at time t .

The components $H_{\alpha\beta}$ of the *surface metric tensor* relative to $\{\mathbf{H}_\alpha\}$ are

$$H_{\alpha\beta} = \mathbf{H}_\alpha \cdot \mathbf{H}_\beta \quad ,$$

and the components $\Omega_{\alpha\beta}$ of the *curvature tensor* relative to $\{\mathbf{H}_\alpha\}$ are

$$\Omega_{\alpha\beta} = \mathbf{N} \cdot \frac{\partial \mathbf{H}_\alpha}{\partial Y^\beta} \quad ;$$

the curvature tensor in the reference configuration will generally have different values from that in the current configuration.

A propagating smooth surface divides the body B into two regions, forming a common boundary between them. The unit normal \mathbf{N} to the surface $S(t)$ is considered to be in the direction in which

$S(t)$ propagates. The region ahead of the surface, that is, the region towards which \mathbf{N} is directed, is denoted by B^+ and the region behind the surface is denoted by B^- . We consider an arbitrary function $f(\mathbf{X}, t)$ which may be scalar-, vector- or tensor-valued, and continuous within both B^+ and B^- . If $S(t)$ is approached from within B^+ and B^- , then f has the definite limits f^+ and f^- . The jump of f at $\mathbf{X} \in S(t)$ is denoted by

$$[f(\mathbf{X})] \equiv f^+(\mathbf{X}) - f^-(\mathbf{X}) .$$

$S(t)$ is called a *singular surface* with respect to f at time t if $[f] \neq 0$. A singular surface that has a non-zero normal velocity, that is,

$$U(\mathbf{X}, t) \neq 0 ,$$

is a *wave*.

Hadamard's Lemma states that for a smooth curve $\ell(s)$ on a singular surface $S(t)$ and a function ϕ which can be scalar-, vector- or tensor-valued, the identity

$$\frac{d}{ds} [\phi] = [\text{Grad } \phi] \cdot \frac{d\ell}{ds} \quad (4.13)$$

holds.

If we take coordinate curves such that $Y^\alpha = \text{constant}$ on $\ell(s)$, then for a scalar-valued function $\phi \equiv z$ (4.13) becomes

$$\frac{\partial}{\partial Y^\alpha} [z] = [\text{Grad } z] \cdot \mathbf{H}_\alpha, \quad (4.14)$$

and for a scalar- or tensor-valued function $\phi \equiv \mathbf{V}$ (4.13) becomes

$$\frac{\partial}{\partial Y^\alpha} [\mathbf{V}] = [\text{Grad } \mathbf{V}] \cdot \mathbf{H}_\alpha, \quad (4.15)$$

where the jumps in the gradients of z and \mathbf{V} are given by (see [84])

$$[\text{Grad } z] = [\text{Grad } z \cdot \mathbf{N}] \mathbf{N} + \sum_{\alpha=1}^2 \frac{\partial}{\partial Y^\alpha} [z] \mathbf{H}^\alpha \quad (4.16)$$

and

$$[\text{Grad } \mathbf{V}] = [(\text{Grad } \mathbf{V}) \mathbf{N}] \otimes \mathbf{N} + \sum_{\alpha=1}^2 \frac{\partial}{\partial Y^\alpha} [\mathbf{V}] \otimes \mathbf{H}^\alpha. \quad (4.17)$$

The results (4.16) and (4.17) are known as the *geometrical conditions of compatibility* (which lead to Maxwell's theorem when $[z] = 0$ or $[\mathbf{V}] = \mathbf{0}$); these identities express the jump of a derivative of a function in terms of the jumps of the normal derivatives, and the tangential derivatives.

The δ -derivative or the displacement derivative $\frac{\delta}{\delta t}$ for a scalar-valued function z is defined by (see [84])

$$\frac{\delta z}{\delta t} = \frac{\partial z}{\partial t} + U(\text{Grad } z) \cdot \mathbf{N} \quad (4.18)$$

and for a vector- or tensor-valued function \mathbf{V} , by

$$\frac{\delta \mathbf{V}}{\delta t} = \frac{\partial \mathbf{V}}{\partial t} + U(\text{Grad } \mathbf{V})\mathbf{N} \quad (4.19)$$

The displacement derivative measures the rate of change of a quantity as seen by an observer moving with normal velocity $U\mathbf{N}$; it is thus the rate of change of the quantity with respect to an observer moving with the normal velocity U of the surface.

If we apply Hadamard's Lemma along a path which has \mathbf{N} as tangent, then (4.18) and (4.19) become

$$\frac{\delta}{\delta t} [z] = [\dot{z}] + U[\text{Grad } z \cdot \mathbf{N}] \quad , \quad (4.20)$$

$$\frac{\delta}{\delta t} [\mathbf{V}] = [\dot{\mathbf{V}}] + U[(\text{Grad } \mathbf{V})\mathbf{N}] \quad , \quad (4.21)$$

which are the *kinematical conditions of compatibility*.

For continuous z and V , that is $[z] = 0$ and $[V] = 0$, equations (4.16) and (4.17) take the forms

$$[\text{Grad } z] = [\text{Grad } z \cdot N]N \quad (4.22)$$

and

$$[\text{Grad } V] = [(\text{Grad } V)N] \otimes N \quad (4.23)$$

Furthermore, equations (4.20) and (4.21) with $[z] = 0$ and $[V] = 0$ become

$$[\text{Grad } z \cdot N] = -U^{-1}[\dot{z}] \quad (4.24)$$

and

$$[(\text{Grad } V)N] = -U^{-1}[\dot{V}] \quad (4.25)$$

If we use \dot{z} instead of z in (4.20) and \dot{V} instead of V in (4.21) we obtain the iterated kinematical *conditions of compatibility*

$$[\text{Grad } \dot{z}] = [(\text{Grad } \dot{z}) \cdot N]N + \sum_{\alpha=1}^2 [\dot{z}]_{,\alpha} H^{\alpha}, \quad (4.26)$$

$$[\ddot{z}] = -U[(\text{Grad } \dot{z})N] + \frac{\delta}{\delta t} [\dot{z}], \quad (4.27)$$

$$[\text{Grad } \dot{V}] = [(\text{Grad } \dot{V})N] \otimes N + \sum_{\alpha=1}^2 [\dot{V}]_{,\alpha} \otimes H^{\alpha}, \quad (4.28)$$

and

$$[\ddot{V}] = -U[(\text{Grad } \dot{V})N] + \frac{\delta}{\delta t} [\dot{V}] \quad (4.29)$$

If we substitute $\text{Grad } z$ and $\text{Grad } V$ instead of z and V , respectively (4.16) and (4.17) we obtain similarly the *iterated geometrical conditions of compatibility*

$$\begin{aligned} [\text{Grad}(\text{Grad } z)] &= [N \cdot \text{Grad}(\text{Grad } z)N]N \otimes N \\ &+ \sum_{\alpha=1}^2 (N \otimes H^{\alpha} + H^{\alpha} \otimes N) [\text{Grad } z \cdot N]_{,\alpha} \\ &- [\text{Grad } z \cdot N] \Omega^{\alpha\beta} H_{\alpha} \otimes H_{\beta} \quad , \end{aligned} \quad (4.30)$$

$$\begin{aligned} [\text{Grad}(\text{Grad } V)] &= [\text{Grad}(\text{Grad } V)(N, N)] \otimes N \otimes N \\ &+ \sum_{\alpha=1}^2 \{[(\text{Grad } V)N]_{,\alpha} \otimes (N \otimes H^{\alpha} + H^{\alpha} \otimes N)\} \\ &- \sum_{\alpha, \beta=1}^2 \Omega_{\alpha\beta} [\text{Grad}(\text{Grad } V)(N, N)] \otimes H^{\alpha} \otimes H^{\beta} \end{aligned} \quad (4.31)$$

for the case in which $[z] = 0$ and $[V] = 0$.

We can obtain the alternative expressions to (4.26) and (4.27) if we use (4.20) with z replaced with \dot{z} ; we find that

$$\begin{aligned}
 [\text{Grad } \dot{z}] &= -U[\mathbf{N} \cdot \text{Grad}(\text{Grad } z)\mathbf{N}]\mathbf{N} + \frac{\delta}{\delta t} [\text{Grad } z \cdot \mathbf{N}]\mathbf{N} \\
 &\quad - \sum_{\alpha=1}^2 (U[\text{Grad } z \cdot \mathbf{N}]_{,\alpha})\mathbf{H}^\alpha
 \end{aligned} \tag{4.32}$$

and

$$\begin{aligned}
 [\ddot{z}] &= U^2[(\mathbf{N} \cdot \text{Grad}(\text{Grad } z)\mathbf{N})] + -2U \frac{\delta}{\delta t} [\text{Grad } z \cdot \mathbf{N}] \\
 &\quad - [\text{Grad } z \cdot \mathbf{N}] \frac{\delta U}{\delta t} .
 \end{aligned} \tag{4.33}$$

Similarly (4.28) and (4.29) become

$$\begin{aligned}
 [\text{Grad } \dot{\mathbf{V}}] &= -U[\text{Grad}(\text{Grad } \mathbf{V})(\mathbf{N},\mathbf{N})] \otimes \mathbf{N} + \frac{\delta}{\delta t} [(\text{Grad } \mathbf{V})\mathbf{N}] \otimes \mathbf{N} \\
 &\quad - \sum_{\alpha=1}^2 U[(\text{Grad } \mathbf{V})\mathbf{N}]_{,\alpha} \otimes \mathbf{H}^\alpha ,
 \end{aligned} \tag{4.34}$$

$$\begin{aligned}
 [\ddot{\mathbf{V}}] &= -U^2[\text{Grad}(\text{Grad } \mathbf{V})(\mathbf{N},\mathbf{N})] - 2U \frac{\delta}{\delta t} [(\text{Grad } \mathbf{V})\mathbf{N}] \\
 &\quad - [(\text{Grad } \mathbf{V})\mathbf{N}] \frac{\delta U}{\delta t} .
 \end{aligned} \tag{4.35}$$

In equations (4.32) and (4.33) we assume $[z] = 0$ and similarly in (4.34) and (4.35) $[V] = 0$. Equations (4.32) - (4.35) are known as Thomas' iterated kinematical conditions of compatibility [24].

Equations (4.16) and (4.17) with $[z] = 0$ and $[V] = 0$ yield the special cases (4.22) and (4.23) of the geometric conditions of capability. Similarly (4.20) and (4.21) take the forms (4.24) and (4.25) for continuous z and V .

Substitution of (4.22) in (4.20), and (4.23) in (4.21) yield the useful identities

$$[\text{Grad } z] = -U^{-1}[\dot{z}]N \quad (4.36)$$

and

$$[\text{Grad } V] = -U^{-1}[\dot{V}] \otimes N \quad (4.37)$$

for continuous z and V .

From (4.37) we can obtain as a special case the result

$$[\text{Div } V] = -U^{-1}[\dot{V}]N \quad (4.38)$$

Acceleration Waves

An acceleration wave is defined to be a propagating singular surface across which displacement $x(X,t)$, velocity $\dot{x}(X,t)$ and deformation gradient $F(X,t)$ are continuous; but quantities such as

acceleration $\ddot{\mathbf{x}}$ and the time rate of deformation gradient $\dot{\mathbf{F}}$ are discontinuous. The lowest ordered discontinuities occur in second derivatives of \mathbf{x} , hence the acceleration wave front is a *second order singular surface*. Higher order derivatives of \mathbf{x} than two are also discontinuous and their discontinuities are expressible in terms of the lower order discontinuities. We assume that all functions which suffer discontinuities across a singular surface are continuous elsewhere in the material body.

We illustrate the profile of an acceleration wave (for a particle) in Figure 4.2 where we denote by x_1 one of the components of \mathbf{x} (which is continuous across the wave front). It is clear that the graph of the variation of \dot{x}_1 with respect to time is also continuous, but at the point $t = t_0$ where it is non-smooth there exists a jump in the corresponding graph of acceleration \ddot{x}_1 . Time t_0 indicates the instant at which the wave front reaches the particle whose motion is being discussed.

The jump in acceleration is known as the *amplitude* and denoted by

$$[\ddot{\mathbf{x}}] = \mathbf{s} \quad . \quad (4.39)$$

If we use equations (4.34) and (4.35) with $\mathbf{V} = \mathbf{x}(\mathbf{X}, t)$ we obtain the jump in the deformation gradient rate

$$[\dot{\mathbf{F}}] = - \mathbf{U}^{-1} \mathbf{s} \otimes \mathbf{N} \quad (4.40)$$

in terms of amplitude and the wave normal.

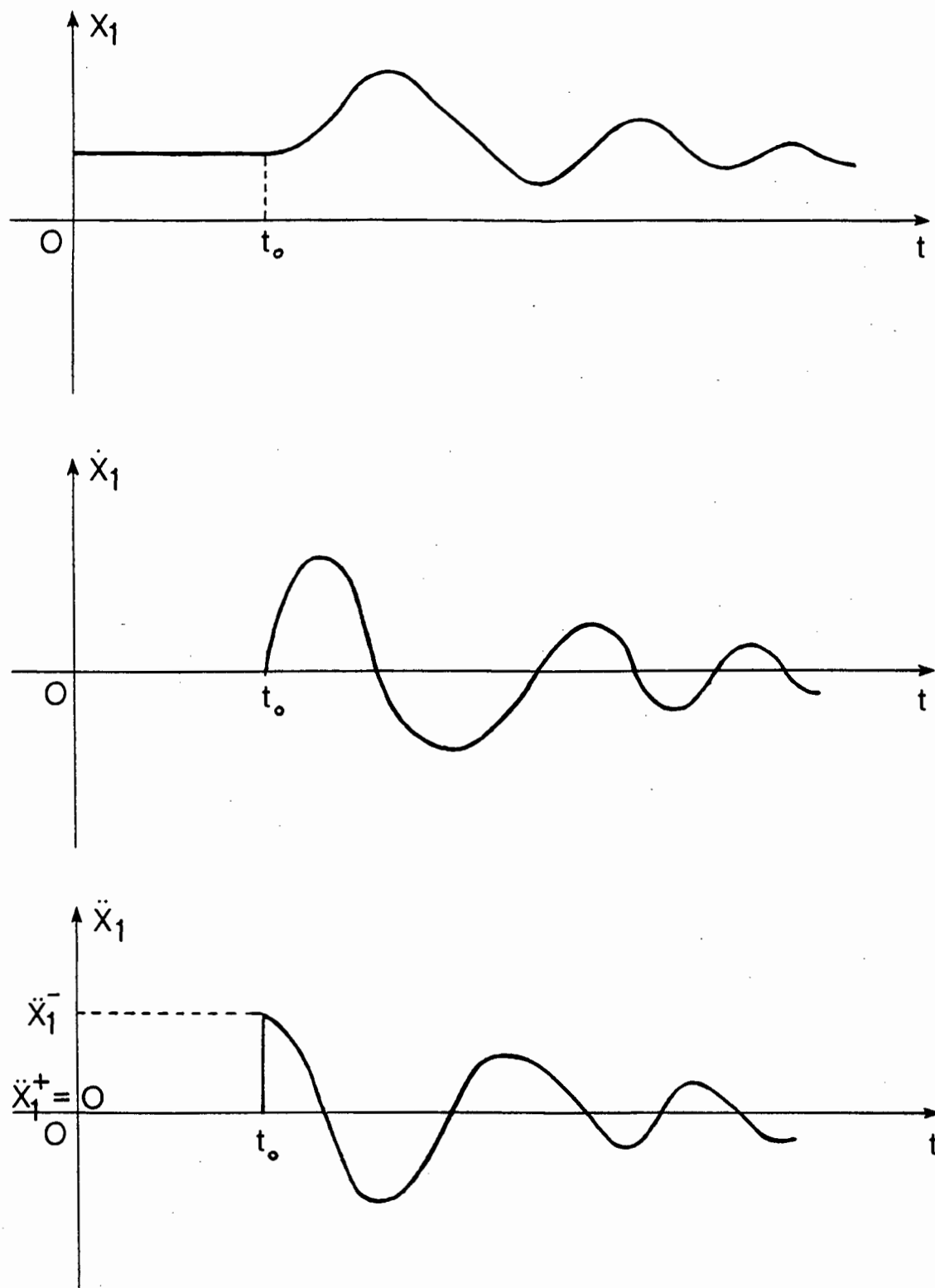


Figure 4.2 The profile of an acceleration wave for x, \dot{x} and \ddot{x} where we assume an arbitrary state for x ; ahead and beyond the wave front.

If we use (4.31) and (4.35) with $V = x(\mathbf{X}, t)$ or (4.37) with $V = \mathbf{F}$ we obtain

$$[\text{Grad } \mathbf{F}] = -U^{-1} [\dot{\mathbf{F}}] \otimes \mathbf{N} = -U^{-2} \mathbf{s} \otimes \mathbf{N} \otimes \mathbf{N} . \quad (4.41)$$

The equation of balance of linear momentum (3.52) in absence of body forces becomes

$$\text{Div } \mathbf{T} = \rho_0 \ddot{\mathbf{x}} ,$$

and the jump of this equation across an acceleration wave front is

$$[\text{Div } \mathbf{T}] = \rho_0 [\ddot{\mathbf{x}}] . \quad (4.42)$$

Equation (4.42) could equally be obtained with a body force $\mathbf{b} \neq 0$, provided that it is assumed to be continuous across the singular surface, that is $[\mathbf{b}] = 0$. If we use (4.38) with $V = \mathbf{T}$ and the definition (4.39), then (4.42) takes the form

$$[\dot{\mathbf{T}}] \mathbf{N} = -\rho_0 U \mathbf{s} \quad (4.43)$$

which is the general form of *the equation of motion* for acceleration waves in any deformable medium.

For the *purely mechanical case* we assume a free energy of the form $\psi = \psi(\mathbf{F})$; using this we obtain the expression for the first

Piola-Kirchhoff stress tensor from (3.55), that is,

$$\mathbf{T} = \rho_0 \frac{\partial \psi}{\partial \mathbf{F}} .$$

The associated stress rate becomes

$$\dot{\mathbf{T}} = \rho_0 \frac{\partial^2 \psi}{\partial \mathbf{F} \partial \mathbf{F}} \dot{\mathbf{F}} . \quad (4.44)$$

If we substitute (4.44) and (4.40) in (4.43) we obtain

$$\mathbf{Q}\mathbf{s} = \rho_0 U^2 \mathbf{s} \Rightarrow (\mathbf{Q} - \rho_0 U^2 \mathbf{I})\mathbf{s} = \mathbf{0} \quad (4.45)$$

which is the *propagation condition*. The acoustic tensor \mathbf{Q} is defined by

$$\mathbf{Q}\mathbf{a} = \rho_0 \frac{\partial^2 \psi}{\partial \mathbf{F} \partial \mathbf{F}} (\mathbf{a} \otimes \mathbf{N})\mathbf{N} \quad \text{for any vector } \mathbf{a} . \quad (4.46)$$

If \mathbf{Q} is symmetric and positive-definite, there exist three real positive proper numbers $\rho_0 U^2$ which give wave speeds in the three directions of proper vectors \mathbf{s} . However, it should be noted that the existence of real and non-zero proper numbers of \mathbf{Q} does not necessarily imply the existence of waves in the material body. The existence of proper numbers of the acoustic tensor is a *necessary but not sufficient* condition for the existence of propagating acceleration waves.

We define *principal waves* as waves travelling in the direction of one of the proper vectors \mathbf{p}_i of \mathbf{Q} , that is

$$\mathbf{N} = \mathbf{p}_i \quad \text{for } i = 1, 2 \text{ or } 3. \quad (4.47)$$

Longitudinal and *transverse waves* are waves fulfilling respectively the conditions

$$\mathbf{s} \times \mathbf{n} = 0 \quad (4.48)$$

and

$$\mathbf{s} \cdot \mathbf{n} = 0 \quad (4.49)$$

The magnitude of amplitude \mathbf{s} , denoted by η , is

$$\eta = \sqrt{\mathbf{s} \cdot \mathbf{s}} \quad (4.50)$$

If we take $i = 1$ in (4.47) in the case of a *longitudinal principal wave* the condition that the amplitude has to satisfy is therefore

$$\mathbf{s} = \eta \mathbf{n} = \eta \mathbf{q}_1 \quad (4.51)$$

Similarly for a *transverse principal wave* we have the condition

$$\mathbf{s} = \eta \mathbf{q}_\Lambda, \quad \Lambda = 2 \text{ or } 3 \quad (4.52)$$

that has to be satisfied by the amplitude.

CHAPTER V

PROPAGATION OF ACCELERATION WAVES IN
ELASTIC-PLASTIC SOLIDS UNDERGOING FINITE STRAINS5.1 INTRODUCTION

The basic aim of this study of acceleration waves in elastic-plastic materials is to reduce the problem of wave propagation to a characteristic value problem similar to (4.45). We do this in this Chapter by deriving the equation of motion of acceleration waves in elastic-plastic solids with the use of the constitutive theory presented in Chapter III and the theory of singular surfaces reviewed in Chapter IV. We first find the propagation condition for acceleration waves in elastic-plastic materials with non-smooth yield surfaces and isotropic hardening, assuming that plastic deformation occurs on both sides of the wave front. We then proceed from the general equation of motion to the special case of a smooth yield surface. For the case of elastic-plastic materials with smooth yield surfaces we compare the possible speeds of propagation in the case of fully plastic loading on both sides of the wave front, that is *plastic wave speeds*, with purely elastic deformations on both sides of the wave front, that is, *elastic wave speeds*, without calculating the actual values of various wave speeds. By certain algebraic manipulations on the propagation condition or on the acoustic tensor we arrive at results analogous to *Mandel's inequalities* in [47,49],

[66] and [3], for the case of materials having a smooth yield surface; these provide a general basis of comparison between the elastic and plastic wave speeds. By making different assumptions on the nature of deformations ahead and behind the wave front we find the propagation conditions for loading waves and unloading waves as well as elastic waves. We find the propagation conditions of principal waves and both longitudinal and transverse principal waves. We exploit the assumption of isotropy and the additional assumption that the elastic behaviour of the materials is neo-Hookean, and compare wave speeds in materials with the von Mises and Tresca yield surfaces.

In contrast to the case of waves in purely elastic materials the equation (4.40) of the jump of $\dot{\mathbf{F}}$ is not adequate to find a propagation condition in plastic materials since the equation of the balance of linear momentum involves the jump of $\dot{\mathbf{F}}^p$ across the wave front, if it is expressed in jump form. Thus the expressibility of the jump of $\dot{\mathbf{F}}^p$ across the wave front in terms of the jump of $\dot{\mathbf{F}}$ is the first major step in finding the acoustic tensor. We explain how we solve this problem and find the general equation of motion of acceleration waves in elastic-plastic materials in Section 5.2. We also present in this Section the reduction of the propagation condition to the corresponding case in which the yield surface is smooth. We find the propagation conditions for plastic waves; for *loading waves*, that is, waves which propagate into a region of unloading and cause a fully plastic state behind the wave front; and for *unloading waves*, that is, waves which propagate into a fully

plastic region, but cause a state of unloading behind the wave front. In these cases of loading and unloading waves we observe a deviation from the characteristic value problem for plastic waves. In Section 5.3 we first derive the propagation condition of *elastic waves*, then compare the sets of elastic and plastic wave speeds by the use of inequalities similar to those of Mandel. In Section 5.4 we derive the propagation conditions for principal waves. Finally, in Section 5.5 we present the comparison of wave speeds in compressible neo-Hookean materials with the von Mises and Tresca yield criteria.

5.2 THE PROPAGATION CONDITION FOR ACCELERATION WAVES

In this Section we derive the equation of motion of acceleration waves assuming different combinations of elastic or plastic states ahead of and behind the wave front. Acceleration waves of three types will be considered:

- (a) Plastic waves: these are waves which propagate into a region which is deformed plastically, and which remains plastically deformed after the passage of the wave front.
- (b) Loading waves: these are waves for which plastic loading occurs behind the wave front, while the material is in a state of unloading ahead of the front.
- (c) Unloading waves: these are waves for which a state of unloading occurs behind the wave front while the region ahead of the front is in a state of plastic deformation.

Any surface in the material body which separates a plastically deformed region from an elastic region, or from a region in which elastic unloading has taken place, is called an *elastic-plastic interface*. Hence, both loading and unloading waves are *moving elastic-plastic interfaces*.

The general equation of motion (4.43) of acceleration waves is valid for any deformable medium, although the time rate $\dot{\mathbf{T}}$ of the first Piola-Kirchhoff stress which appears in (4.43) has to be written explicitly; that depends on the constitutive theory of the particular medium under consideration. We will use in particular the material presented in Section 6 of Chapter III where we derived an expression for the convected rate form $\dot{\bar{\sigma}}$ of Kirchhoff stress.

Using (3.49) and (3.50) we obtain the relation

$$\mathbf{T} = \bar{\sigma} \mathbf{F}^{-T} \quad (5.1)$$

between \mathbf{T} and $\bar{\sigma}$, from which we can obtain the rate

$$\dot{\mathbf{T}} = \dot{\bar{\sigma}} \mathbf{F}^{-T} - \bar{\sigma} \mathbf{L}^T \mathbf{F}^{-T} \quad , \quad (5.2)$$

whose jump across the singular surface of an acceleration wave becomes

$$[\dot{\mathbf{T}}] = [\dot{\bar{\sigma}}] \mathbf{F}^{-T} - \bar{\sigma} [\mathbf{L}^T] \mathbf{F}^{-T} \quad . \quad (5.3)$$

If we substitute (5.3) in (4.43) we obtain

$$[\dot{\bar{\sigma}}] \mathbf{F}^{-T} \mathbf{N} - \bar{\sigma} [\mathbf{L}^T] \mathbf{F}^{-T} \mathbf{N} = -\rho_0 U \mathbf{s}$$

or

$$[\dot{\bar{\sigma}}] \mathbf{n} - \bar{\sigma} [\mathbf{L}^T] \mathbf{n} = -\frac{\rho_0}{\ell} U \mathbf{s} \quad (5.4)$$

by using the relation (4.12), where

$$\ell = |\mathbf{F}^{-T} \mathbf{N}| \quad .$$

We substitute (3.119) in (5.4) to obtain

$$([\dot{\bar{\sigma}}] - \bar{\sigma} [\mathbf{L}^T] + [\mathbf{W}] \bar{\sigma} - \bar{\sigma} [\mathbf{W}]) \mathbf{n} = -\frac{\rho_0}{\ell} U \mathbf{s} \quad . \quad (5.5)$$

The jump $[\dot{\bar{\sigma}}]$ of the convected rate of Kirchhoff stress can be written in terms of the jumps $[\lambda_\beta]$ of the scalar parameters λ_β and the jump $[\mathbf{D}]$ of the deformation rate \mathbf{D} using (3.121) and (3.109), and is

$$[\dot{\bar{\sigma}}] = \mathcal{L}[\mathbf{D}] - \sum_{\beta=1}^m [\lambda_\beta] \mathcal{L}(\mathbf{M}^\beta) \quad . \quad (5.6)$$

Now we need to find the jumps $[L^T]$, $[D]$ and $[W]$. If we substitute (4.40) in (3.34) we obtain the jump

$$[L] = -\ell U^{-1} s \otimes n \quad (5.7)$$

of the velocity gradient and

$$[L^T] = -\ell U^{-1} n \otimes s, \quad (5.8)$$

of its transpose. Substitution of (5.7) and (5.8) in (3.38) yields the jump

$$[D] = -\frac{1}{2} \ell U^{-1} (s \otimes n + n \otimes s) \quad (5.9)$$

of the deformation rate, and the jump

$$[W] = -\frac{1}{2} \ell U^{-1} (s \otimes n - n \otimes s) \quad (5.10)$$

of the spin tensor W is similarly obtained.

If we substitute (5.6), (5.8), (5.9) and (5.10) in (5.5) the equation of motion of acceleration waves in elastic-plastic materials

now becomes

$$\left\{ \mathbf{E} + \frac{1}{2} (\mathbf{t} \cdot \mathbf{n}) \mathbf{I} - \frac{1}{2} \bar{\sigma} - \frac{1}{2} (\mathbf{t} \otimes \mathbf{n} + \mathbf{n} \otimes \mathbf{t}) \right\} \mathbf{s} \\ + \frac{U}{\ell} \sum_{\alpha=1}^m [\lambda_{\alpha}] \mathcal{L}(\mathbf{H}^{\alpha}) \mathbf{n} = \frac{\rho_0}{\ell^2} U^2 \mathbf{s} \quad (5.11)$$

where the components of \mathbf{E} are

$$E_{ik} = \mathcal{L}_{ijkl} n_j n_l, \quad (5.12)$$

and of \mathbf{t} are

$$t_i = \bar{\sigma}_{ij} n_j. \quad (5.13)$$

Plastic Waves

Since for plastic waves plastic deformation occurs on both sides of the wave front we have the conditions

$$\mathbf{D}^{p^+} \neq 0, \quad \mathbf{D}^{p^-} \neq 0 \quad (5.14)$$

and

$$\lambda_{\beta}^+ > 0, \quad \lambda_{\beta}^- > 0$$

ahead and behind the singular surface, for each active value of β .

We obtain the jump

$$[\lambda_\beta] = \sum_{\alpha=1}^m G_{\alpha\beta}^{-1} \mathbf{M}^\alpha \cdot \mathcal{L}[\mathbf{D}] \quad (5.15)$$

of λ_β from (3.130). Now, by using (5.15) with (5.9) in (5.11) we can transform (5.11) to the form

$$\mathbf{Q}\mathbf{s} = \frac{\rho_0}{\ell^2} U^2 \mathbf{s} \quad (5.16)$$

which is a characteristic value problem; the *elastic-plastic acoustic tensor* \mathbf{Q} is given by

$$\begin{aligned} \mathbf{Q} = & \mathbf{E} + \frac{1}{2} (\mathbf{t} \cdot \mathbf{n}) \mathbf{I} - \frac{1}{2} \bar{\sigma} - \frac{1}{2} (\mathbf{t} \otimes \mathbf{n} + \mathbf{n} \otimes \mathbf{t}) \\ & - G_{\alpha\beta}^{-1} \mathbf{a}^\alpha \otimes \mathbf{a}^\beta \end{aligned} \quad (5.17)$$

where the vectors \mathbf{a}^α are defined by

$$\mathbf{a}^\alpha = \mathcal{L}(\mathbf{M}^\alpha) \mathbf{n} \quad (5.18)$$

We note from (5.17) the important property that \mathbf{Q} is *symmetric*.

We obtain the three positive proper numbers $\nu_i = \frac{\rho_0}{\ell^2} U_i^2$ from the

characteristic equation

$$\det (\mathbf{Q} - \nu \mathbf{I}) = 0$$

if \mathbf{Q} is positive-definite.

For an elastic-plastic material with a smooth yield surface the acoustic tensor becomes

$$\begin{aligned} \mathbf{Q} = & \mathbf{E} + \frac{1}{2} (\mathbf{t} \cdot \mathbf{n}) \mathbf{I} - \frac{1}{2} \bar{\sigma} - \frac{1}{2} (\mathbf{t} \otimes \mathbf{n} + \mathbf{n} \otimes \mathbf{t}) \\ & - G^{-1} \mathbf{a} \otimes \mathbf{a} \quad , \end{aligned} \quad (5.19)$$

where $m = a = \beta = 1$, $\mathbf{a} \equiv \mathbf{a}^1$, and $G^{-1} \equiv G_{11}^{-1}$.

Loading Waves

For loading waves the region ahead of the wave front is in a state of unloading, so we have the conditions

$$\mathbf{D}^{p+} = 0 \quad (5.20)$$

and

$$\dot{f}^{\alpha+} < 0 \quad , \quad \lambda_{\alpha}^{+} = 0 \quad , \quad \text{for each } \alpha \quad , \quad (5.21)$$

for this region. For the region behind the wave front where plastic loading occurs the corresponding conditions are

$$\mathbf{D}^{p-} \neq 0 \quad , \quad (5.22)$$

and

$$\dot{f}^{\alpha-} = 0 \quad , \quad \lambda_{\alpha}^{-} > 0 \quad , \quad \text{for each active value of } \alpha \quad . \quad (5.23)$$

Using (5.22) and (5.23) we obtain the jump

$$[\lambda_{\alpha}] = - \lambda_{\alpha}^{-} \quad (5.24)$$

of λ_{α} across the loading wave front.

Substitution of (5.24) in (5.11) yields

$$\left\{ \mathbf{E} + \frac{1}{2} (\mathbf{t} \cdot \mathbf{n}) \mathbf{I} - \frac{1}{2} \bar{\sigma} - \frac{1}{2} (\mathbf{t} \otimes \mathbf{n} + \mathbf{n} \otimes \mathbf{t}) - \ell^{-1} U \sum_{\beta=1}^m \lambda_{\beta}^{-} \mathcal{L}(\mathbf{M}^{\beta}) \mathbf{n} \right\} \mathbf{s} = \frac{\rho_0}{\ell^2} U^2 \mathbf{s} \quad . \quad (5.25)$$

We obtain the expression

$$\lambda_{\beta}^{-} = \sum_{\alpha=1}^m G_{\alpha\beta}^{-1} \mathbf{M}^{\alpha} \cdot \mathcal{L}(\mathbf{D}^{-}) \quad (5.26)$$

from (3.130), and if we substitute $\mathbf{D}^- = \mathbf{D}^+ - [\mathbf{D}]$ in (5.26) it becomes

$$\lambda_{\beta}^- = \sum_{\alpha=1}^m G_{\alpha\beta}^{-1} \mathbf{M}^{\alpha} \cdot (\mathcal{L}(\mathbf{D}^+) - \mathcal{L}[\mathbf{D}]) \quad (5.27)$$

Now, we can substitute (5.27) in (5.25) and obtain the propagation condition

$$\left\{ \mathbf{E} + \frac{1}{2} (\mathbf{t} \cdot \mathbf{n}) \mathbf{I} - \frac{1}{2} \bar{\sigma} - \frac{1}{2} (\mathbf{t} \otimes \mathbf{n} + \mathbf{n} \otimes \mathbf{t}) - G_{\alpha\beta}^{-1} (\mathbf{a}^{\alpha} \otimes \mathbf{a}^{\beta}) \right\} \mathbf{s} - \ell^{-1} U G_{\alpha\beta}^{-1} (\mathbf{P}^{\alpha} \cdot \mathbf{D}^+) \mathbf{a}^{\beta} = \frac{\rho_0 U^2}{\ell^2} \mathbf{s} \quad (5.28)$$

of loading waves, where

$$\mathbf{P}^{\alpha} = \mathcal{L}(\mathbf{M}^{\alpha}) \quad \text{or} \quad P_{ij}^{\alpha} = \mathcal{L}_{ijkl} M_{kl}^{\alpha} \quad (5.29)$$

Equation (5.28) is no longer a characteristic value problem because of the term involving \mathbf{D}^+ on the left hand side of the equation. This result is compatible with the results of Balaban, Green and Naghdi [3] whereas it differs from the results of Raniecki [66], where the scalar parameter λ^- is expressed as a function of the jump $[\epsilon]$ of strain in the context of small-strain plasticity. This is contrary to (5.26) where λ_{β}^- is to be expressed as a function of \mathbf{D}^- rather than that of $[\mathbf{D}]$ which leads to an expression of λ_{β}^- in terms of ϵ^- in a parallel treatment in the context of small-strain

elastoplasticity. The expression of λ as a function of $[\epsilon]$ is effected with the aid of an assumption about the magnitude of the scalar parameter ν [66] defined by

$$\nu = \frac{\partial \hat{f}}{\partial \sigma} \cdot \dot{\sigma}^+ / \frac{\partial \hat{f}}{\partial \sigma} \cdot \dot{\sigma}^-$$

where $\hat{f} = f(\sigma)$ is the yield function. The scalar parameter ν is in fact associated with the jump $\left[\frac{\partial \hat{f}}{\partial \sigma} \cdot \dot{\sigma} \right] = \frac{\partial \hat{f}}{\partial \sigma} \cdot [\dot{\sigma}]$, which is unknown.

If we assume that the region ahead of the wave front is at rest, so that $\mathbf{D}^+ = 0$ in (5.28), we obtain equation (5.16) for plastic waves with the acoustic tensor (5.17).

Unloading Waves

For this case the region ahead of the wave front is in a state of plastic deformation, so that

$$\mathbf{D}^{p+} \neq 0$$

and

$$\dot{f}^{\alpha+} = 0, \lambda_{\alpha}^+ > 0, \text{ for active values } \alpha. \quad (5.30)$$

The region behind the wave front is in a state of unloading, so that here

$$\mathbf{D}^{\mathbf{p}^-} = 0$$

and

$$\dot{f}^{\alpha^-} < 0, \lambda_{\alpha}^- = 0, \text{ for each } \alpha. \quad (5.31)$$

Equations (5.31) and (5.30) yield

$$[\lambda_{\alpha}] = \lambda_{\alpha}^+. \quad (5.32)$$

and (3.130) gives

$$\lambda_{\beta}^+ = \sum_{\alpha=1}^m G_{\alpha\beta}^{-1} \mathbf{M}^{\alpha} \cdot \mathcal{L}(\mathbf{D}^+) \quad (5.33)$$

Substitution of (5.32) and (5.33) in (5.11) yields

$$\left\{ \mathbf{E} + \frac{1}{2} (\mathbf{t} \cdot \mathbf{n}) \mathbf{I} - \frac{1}{2} \bar{\sigma} - \frac{1}{2} (\mathbf{t} \otimes \mathbf{n} + \mathbf{n} \otimes \mathbf{t}) \right. \\ \left. + \ell^{-1} U G_{\alpha\beta}^{-1} \{ \mathbf{M}^{\alpha} \cdot \mathcal{L}(\mathbf{D}^+) \} \mathbf{a}^{\beta} \right\} \mathbf{s} = \frac{\rho_0 U^2}{\ell^2} \mathbf{s} \quad (5.34)$$

which is the propagation condition of unloading waves; this is no longer a characteristic value problem, as in the case of loading waves.

5.3 COMPARISON OF ELASTIC AND PLASTIC WAVE SPEEDS

In this Section we first derive the propagation condition for elastic waves, that is, the propagation condition for acceleration waves in purely elastic media, and seek an *elastic acoustic tensor* similar to (4.46), from the general equation of motion (5.11) of acceleration waves in elastic-plastic media. We assume the existence of three distinct proper numbers, or the *elastic wave speeds* corresponding to the elastic acoustic tensor, and then compare these with the *plastic wave speeds*, or the proper numbers of the *elastic-plastic acoustic tensor* (5.17). The results in this Section are confined to the case of smooth yield surfaces.

Elastic Waves

The condition of the occurrence of elastic waves is the existence of purely elastic deformations both behind and ahead of the acceleration wave front; that is,

$$\mathbf{D}^{\mathbf{p}^+} = 0 \quad , \quad \lambda_{\alpha}^+ = 0$$

and

$$\mathbf{D}^{\mathbf{p}^-} = 0 \quad , \quad \lambda_{\alpha}^- = 0 \quad ,$$

which yields

$$[\lambda^{\alpha}] = 0 \quad , \quad \text{for each } \alpha \quad . \quad (5.35)$$

If we substitute (5.35) in (5.31) we obtain the propagation condition

$$\left\{ \mathbf{E} + \frac{1}{2} (\mathbf{t} \cdot \mathbf{n}) \mathbf{I} - \frac{1}{2} \bar{\sigma} - \frac{1}{2} (\mathbf{t} \otimes \mathbf{n} + \mathbf{n} \otimes \mathbf{t}) \right\} \mathbf{s} = \frac{\rho_0}{\ell^2} U^2 \mathbf{s} \quad (5.36)$$

for elastic waves, in which case the (symmetric) elastic acoustic tensor \mathbf{Q}^e is

$$\mathbf{Q}^e = \mathbf{E} + \frac{1}{2} (\mathbf{t} \cdot \mathbf{n}) \mathbf{I} - \frac{1}{2} \bar{\sigma} - \frac{1}{2} (\mathbf{t} \otimes \mathbf{n} + \mathbf{n} \otimes \mathbf{t}) \quad . \quad (5.37)$$

Hence, for the general case we can obtain an additive decomposition of the elastic-plastic tensor \mathbf{Q} (5.17) into its elastic and plastic parts, in the form

$$\mathbf{Q} = \mathbf{Q}^e + \mathbf{Q}^p \quad (5.38)$$

where the plastic part \mathbf{Q}^p is

$$\mathbf{Q}^p = - G_{\alpha\beta}^{-1} \mathbf{a}^\alpha \otimes \mathbf{a}^\beta \quad . \quad (5.39)$$

In the case of a smooth yield surface, (5.39) becomes

$$\mathbf{Q}^p = - G^{-1} \mathbf{a} \otimes \mathbf{a} \quad . \quad (5.40)$$

If \mathbf{Q}^e is positive-definite it has three proper numbers ν_i which we order by

$$\nu_1 \geq \nu_2 \geq \nu_3 \quad . \quad (5.41)$$

The corresponding elastic wave speeds are similarly ordered by

$$U_1^e \geq U_2^e \geq U_3^e \quad (U_i^e = \ell \sqrt{\nu_i / \rho_0}) \quad . \quad (5.42)$$

Mandel's Inequalities

We confine attention to the case of *plastic* or *loading* waves with the region ahead of the wave at rest.

If we use (5.37) and (5.40) the propagation condition (5.16) becomes, for the case of a smooth yield surface,

$$(\mathbf{Q}^e - \mathbf{G}^{-1} \mathbf{a} \otimes \mathbf{a} - \nu \mathbf{I}) \mathbf{s} = 0 \quad . \quad (5.43)$$

Hence, the characteristic polynomial equation is

$$F(\nu) = \det (\mathbf{\Lambda}^e - \mathbf{G}^{-1} \mathbf{a} \otimes \mathbf{a}) = 0 \quad (5.44)$$

where

$$\mathbf{\Lambda}^e = \mathbf{Q}^e - \nu \mathbf{I} \quad . \quad (5.45)$$

Equation (5.44) has the alternative expression

$$\begin{aligned} F(\nu) &= \det \{ \mathbf{\Lambda}^e (\mathbf{I} - \mathbf{G}^{-1} \mathbf{\Lambda}^{e^{-1}} \mathbf{a} \otimes \mathbf{a}) \} \\ &= (\det \mathbf{\Lambda}^e) \det (\mathbf{I} - \mathbf{G}^{-1} \mathbf{\Lambda}^{e^{-1}} \mathbf{a} \otimes \mathbf{a}) = 0 \quad . \end{aligned} \quad (5.46)$$

If we use the identity

$$\det (\mathbf{I} - c \mathbf{a} \otimes \mathbf{b}) = 1 - c \mathbf{a} \cdot \mathbf{b} \quad , \quad (5.47)$$

which holds for all vectors \mathbf{a} and \mathbf{b} and scalars c , then (5.47) becomes

$$\begin{aligned} F(\nu) &= \det \mathbf{\Lambda}^e - G^{-1} (\det \mathbf{\Lambda}^e) [\mathbf{\Lambda}^e]^{-1} \mathbf{a} \cdot \mathbf{a} \\ &= \det \mathbf{\Lambda}^e - G^{-1} \mathbf{a} \cdot \left[\frac{\partial (\det \mathbf{\Lambda}^e)}{\partial \mathbf{\Lambda}^e} \mathbf{a} \right] = 0 \quad , \end{aligned} \quad (5.48)$$

since $\frac{\partial \det \mathbf{\Lambda}^e}{\partial \mathbf{\Lambda}^e} = (\det \mathbf{\Lambda}^e) \mathbf{\Lambda}^{e-T} = (\det \mathbf{\Lambda}^e) \mathbf{\Lambda}^{e-1}$, making use of the symmetry of $\mathbf{\Lambda}^e$.

If we let the coordinate axes coincide with the principal axes of \mathbf{Q}^e , (and hence also of $\mathbf{\Lambda}^e$) , then from the first line of (5.48) and (5.45) we obtain

$$\begin{aligned} F(\nu) &= (Q_1^e - \nu)(Q_2^e - \nu)(Q_3^e - \nu) \\ &\quad - G^{-1} \left\{ (Q_2^e - \nu)(Q_3^e - \nu)a_1^2 + (Q_3^e - \nu)(Q_1^e - \nu)a_2^2 \right. \\ &\quad \left. + (Q_1^e - \nu)(Q_2^e - \nu)a_3^2 \right\} = 0 \end{aligned} \quad (5.49)$$

where Q_1^e , Q_2^e and Q_3^e are the proper numbers of Q^e and a_1 , a_2 , a_3 are the components of \mathbf{a} .

In accordance with (5.41) we write

$$Q_1^e \geq Q_2^e \geq Q_3^e \quad \text{for } G^{-1} > 0, \quad (5.50)$$

and consider the following values of the polynomial $F(\nu)$ using (5.49):

$$\begin{aligned} F(Q_1^e) &\leq 0, \\ F(Q_2^e) &\geq 0, \\ F(Q_3^e) &\leq 0, \\ F(\nu) &< 0 \quad \text{for } \nu > Q_1^e. \end{aligned} \quad (5.51)$$

We illustrate the graph F versus ν in Figure 5.1, where we indicate the relative values of the proper numbers Q_i^e of the elastic acoustic tensor Q^e and the proper numbers ν_i of the elastic-plastic acoustic tensor Q .

Using (5.51) we conclude, in view of (5.50), that

$$Q_1^e \geq \nu_1 \geq Q_2^e \geq \nu_2 \geq Q_3^e \geq \nu_3 \quad (5.52)$$

which implies that

$$U_1^e \geq U_1^p \geq U_2^e \geq U_2^p \geq U_3^e \geq U_3^p \quad (5.53)$$

where U_i^p denote the plastic wave speed corresponding to ν_i ($i = 1, 2, 3$) .

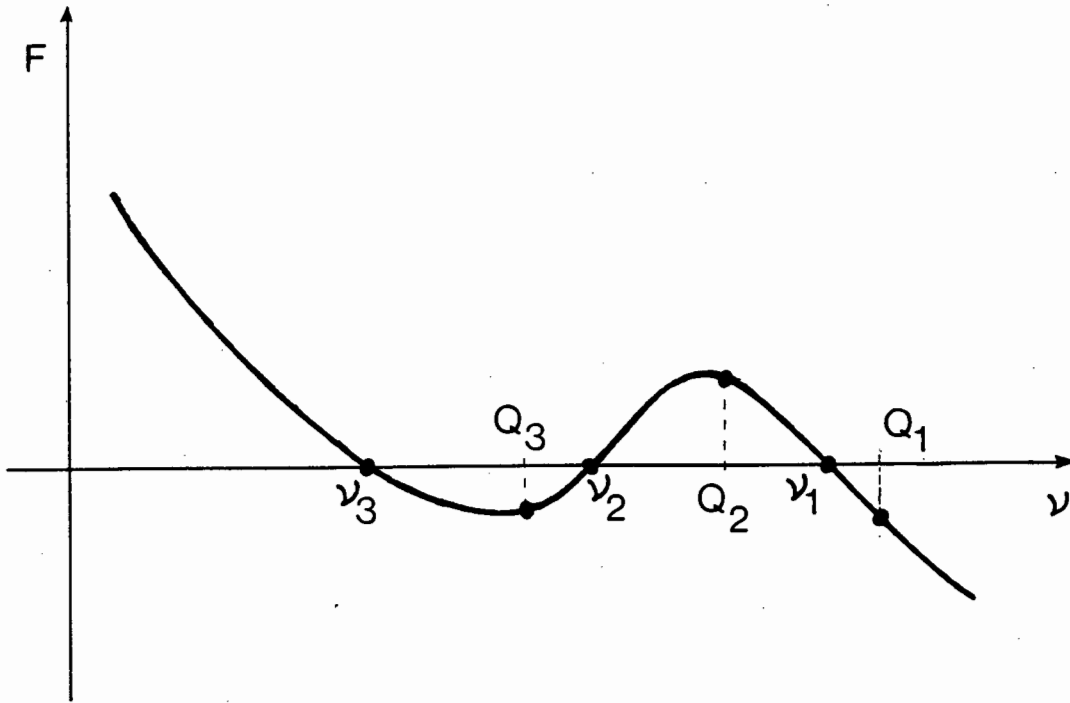


Figure 5.1 The graph of an arbitrary characteristic polynomial with the roots ν_1, ν_2, ν_3 .

5.4 PRINCIPAL WAVES

In this Section we derive the propagation conditions for principal waves, longitudinal waves and transverse waves, then find the expressions of relevant speeds of propagation. In order to do this, we first need to find the spectral representation of the elastoplastic acoustic tensor in terms of the fourth order tensor \mathcal{L}

on which it depends, as well as its additive components. This study is confined to *isotropic* materials, and we will assume furthermore that the region ahead of the wave is *at rest*. Since it is not possible to obtain results of general for waves propagating in bodies which are subject to arbitrary states of stress, we confine the study to the situation in which the body is in a state of *uniaxial stress*.

Using (3.69) and (3.116) we define \mathbf{B} in an alternative form by

$$\mathbf{B} = \frac{\partial \mathbf{S}^e}{\partial \mathbf{C}^e} \quad . \quad (5.55)$$

For an isotropic material the free energy depends symmetrically on the proper numbers c_i^e of \mathbf{C}^e , where $c_i^e = (a_i^e)^2$.

From (3.69) and (3.27) we therefore find that the stress \mathbf{S}^e has the spectral representation

$$\mathbf{S}^e = 2\rho_0 \sum_{i=1}^3 \frac{\partial \psi}{\partial c_i^e} \beta_i \otimes \beta_i \quad . \quad (5.56)$$

We express the time rate $\dot{\mathbf{S}}^e$ of \mathbf{S}^e by

$$\dot{\mathbf{S}}^e = \mathbf{B}(\dot{\mathbf{C}}^e) \quad , \quad (5.57)$$

using (3.69) and (5.55).

The spectral representation of \mathbf{B} is obtained using (5.55), (5.56) and (5.57) (see [16] for a derivation), and is

$$\begin{aligned} \mathbf{B} = & \sum_{i,j=1}^3 \frac{\partial S_i^e}{\partial c_j^e} \beta_i \otimes \beta_i \otimes \beta_j \otimes \beta_j \\ & + \sum_{\substack{i,j=1 \\ i \neq j \\ c_i^e \neq c_j^e}}^3 \frac{1}{2} \frac{S_i^e - S_j^e}{c_i^e - c_j^e} (\beta_i \otimes \beta_j \otimes \beta_i \otimes \beta_j + \beta_i \otimes \beta_j \otimes \beta_j \otimes \beta_i) . \end{aligned} \quad (5.58)$$

Using (5.58) and (3.31) in (3.118) we obtain the spectral representation

$$\begin{aligned} \xi = & \sum_{i,j=1}^3 d_{ij} \mathbf{q}_i \otimes \mathbf{q}_i \otimes \mathbf{q}_j \otimes \mathbf{q}_j \\ & + \sum_{\substack{i,j=1 \\ i \neq j}}^3 e_{ij} (\mathbf{q}_i \otimes \mathbf{q}_j \otimes \mathbf{q}_i \otimes \mathbf{q}_j + \mathbf{q}_i \otimes \mathbf{q}_j \otimes \mathbf{q}_j \otimes \mathbf{q}_i) \end{aligned} \quad (5.59)$$

of ξ , where

$$d_{ij} = 2B_{iijj} c_i^e c_j^e = 2 \frac{\partial S_i^e}{\partial c_j^e} c_i^e c_j^e \quad (5.60)$$

and

$$e_{ij} = 2B_{ijij} c_i^e c_j^e = 2B_{ijji} c_i^e c_j^e = \frac{S_i^e - S_j^e}{c_i^e - c_j^e} c_i^e c_j^e \quad . \quad (5.61)$$

In the event that $c_j^e = c_i^e$, e_{ij} is obtained by a limiting procedure [16]. These details are not presented since, as will be seen, terms containing e_{ij} in the propagation condition will fall away.

The spectral representation of $\bar{\sigma}$ is

$$\bar{\sigma} = 2\rho_0 \sum_{i=1}^3 c_i^e \frac{\partial \psi}{\partial c_i^e} \mathbf{q}_i \otimes \mathbf{q}_i \quad , \quad (5.62)$$

using (5.56), (3.70) and (3.31).

Therefore, the spectral representation of \mathcal{L} can now be obtained from (3.122), and is

$$\mathcal{L} = \xi + \sum_{i,j=1}^3 (\bar{\sigma}_i + \bar{\sigma}_j) \mathbf{q}_i \otimes \mathbf{q}_j \otimes \mathbf{q}_i \otimes \mathbf{q}_j \quad (5.63)$$

where

$$\bar{\sigma}_i = 2\rho_0 c_i^e \frac{\partial \psi}{\partial c_i^e} \quad (5.64)$$

and ξ is given by (5.59). We obtain the spectral representation of \mathbf{E} whose components are given in (5.12), in the form

$$\begin{aligned} \mathbf{E} = & \sum_{i,j=1}^3 d_{ij} (\mathbf{q}_i \cdot \mathbf{n})(\mathbf{q}_j \cdot \mathbf{n}) \mathbf{q}_i \otimes \mathbf{q}_j \\ & + \sum_{\substack{i,j=1 \\ i \neq j}}^3 e_{ij} (\mathbf{q}_j \cdot \mathbf{n})^2 \mathbf{q}_i \otimes \mathbf{q}_i + \sum_{i,j=1}^3 (\bar{\sigma}_i + \bar{\sigma}_j) (\mathbf{q}_j \cdot \mathbf{n})^2 \mathbf{q}_i \otimes \mathbf{q}_i . \end{aligned} \quad (5.65)$$

The traction vector \mathbf{t} in (5.13) becomes

$$\mathbf{t} = \sum_{i=1}^3 \bar{\sigma}_i (\mathbf{q}_i \cdot \mathbf{n}) \mathbf{q}_i . \quad (5.66)$$

The yield functions $f_0^\alpha(\bar{\sigma})$ are assumed to be isotropic functions of the stress so that, from (3.111),

$$\mathbf{M}^\alpha = \sum_{i=1}^3 \mathbb{M}_i^\alpha \mathbf{q}_i \otimes \mathbf{q}_i . \quad (5.67)$$

Using (5.67) and (5.63) in (5.18) we obtain

$$\mathbf{a}^\alpha = \sum_{i,j=1}^3 (\mathbf{q}_i \cdot \mathbf{n}) (d_{ij} \mathbf{M}_j^\alpha + 2\mathbf{M}_i^\alpha \bar{\sigma}_i) \mathbf{q}_i \quad . \quad (5.68)$$

Using (5.65), (5.67) and (5.68) in (5.17), and assuming that

$$\mathbf{n} = \mathbf{q}_3 \quad (5.69)$$

we obtain the propagation condition for *principal waves* in the form

$$\left[d_{33} \mathbf{q}_3 \otimes \mathbf{q}_3 + \sum_{i=1}^3 (\bar{\sigma}_i + \bar{\sigma}_3) \mathbf{q}_i \otimes \mathbf{q}_i + \sum_{i=1}^2 e_{i3} \mathbf{q}_i \otimes \mathbf{q}_i \right. \\ \left. + \frac{1}{2} \bar{\sigma}_3 \mathbf{I} - \frac{1}{2} \bar{\sigma} - \bar{\sigma}_3 \mathbf{q}_3 \otimes \mathbf{q}_3 \right. \\ \left. - G_{\alpha\beta}^{-1} \mathbf{A}^\alpha \mathbf{A}^\beta \mathbf{q}_3 \otimes \mathbf{q}_3 \right] \mathbf{s} = \frac{\rho_0 U^2}{\ell^2} \mathbf{s} \quad (5.70)$$

where

$$\mathbf{A}^\alpha = d_{3j} \mathbf{M}_j^\alpha + 2\mathbf{M}_3^\alpha \bar{\sigma}_3 \quad (5.71)$$

and

$$\mathbf{a}^\alpha = \mathbf{A}^\alpha \mathbf{q}_3$$

for the case in which $\mathbf{n} = \mathbf{q}_3$.

Longitudinal Waves

The condition for the existence of longitudinal principal waves is that \mathbf{n} and the acceleration jump \mathbf{s} are equidirectional. Therefore (4.51) becomes

$$\mathbf{s} = \eta \mathbf{q}_3 \quad . \quad (5.72)$$

If we substitute (5.72) in (5.70) we obtain the propagation condition

$$d_{33} \mathbf{q}_3 + (\bar{\sigma}_1 + \bar{\sigma}_3) \delta_{i3} \mathbf{q}_i \quad (5.73)$$

$$- \bar{\sigma}_3 \mathbf{q}_3 - G_{\alpha\beta}^{-1} A^\alpha A^\beta \mathbf{q}_3 = \frac{\rho_0 U_L^2}{\ell^2} \mathbf{q}_3$$

where U_L is the speed of propagation of longitudinal waves. It follows that

$$\frac{\rho_0 U_L^2}{\ell^2} = d_{33} + \bar{\sigma}_3 - G_{\alpha\beta}^{-1} A^\alpha A^\beta \quad . \quad (5.74)$$

Since $d_{33} > 0$ and $G_{\alpha\beta}^{-1} A^\alpha A^\beta > 0$ a necessary condition for the existence of real wave speeds is therefore that

$$d_{33} + \bar{\sigma}_3 > G_{\alpha\beta}^{-1} A^\alpha A^\beta \quad . \quad (5.75)$$

Transverse Waves

In the case of transverse principal waves condition (4.49) has to be satisfied, that is, the wave propagates in one of the principal directions but the jump s in acceleration is in one of the remaining principal directions. We write

$$\mathbf{s} = \eta \mathbf{q}_\Gamma \quad (5.76)$$

where $\Gamma = 1$ or 2 , since we assume that the wave propagates in the direction \mathbf{q}_3 , as before.

Therefore (4.49) becomes

$$\mathbf{q}_3 \cdot \mathbf{q}_\Gamma = 0 \quad .$$

If we substitute (5.76) in (5.70) we obtain

$$(\bar{\sigma}_\Gamma + \bar{\sigma}_3) \mathbf{q}_\Gamma + e_{\Gamma 3} \mathbf{q}_\Gamma + \frac{1}{2} \bar{\sigma}_3 \mathbf{q}_\Gamma$$

$$- \frac{1}{2} \sigma_\Gamma \mathbf{q}_\Gamma = \frac{\rho_0 U_T^2}{\ell^2} \mathbf{q}_\Gamma$$

where U_T is the speed of propagation of transverse waves. Thus

$$\frac{\rho_0 U_T^2}{\ell^2} = \frac{3}{2} \bar{\sigma}_3 + \frac{1}{2} \bar{\sigma}_\Gamma + e_{\Gamma 3} \quad . \quad (5.77)$$

It is especially noteworthy that *the propagation condition for transverse waves is independent of plastic behaviour.*

5.5 COMPARISON OF WAVE SPEEDS IN ELASTIC-PLASTIC MATERIALS WITH VON MISES AND TRESCA YIELD CRITERIA

In this Section we compare the speeds of propagation in *isotropic* elastic-plastic materials with the von Mises and Tresca yield criteria, using a free energy function for compressible neo-Hookean materials. The assumption of isotropy enables us to express the yield functions in terms of principal stresses and the free energy in terms of elastic principal stretches.

Yield Surfaces for Materials with von Mises and Tresca Yield Conditions

The assumption of isotropy implies that the yield function f can be expressed in the form

$$f = f(\bar{\sigma}_1, \bar{\sigma}_2, \bar{\sigma}_3)$$

where $\bar{\sigma}_1, \bar{\sigma}_2$ and $\bar{\sigma}_3$ are the principal Kirchhoff stresses. The von Mises yield surface in principal stress space is the cylinder defined by

$$\sqrt{(\bar{\sigma}_1 - \bar{\sigma}_2)^2 + (\bar{\sigma}_2 - \bar{\sigma}_3)^2 + (\bar{\sigma}_3 - \bar{\sigma}_1)^2} - \frac{k}{\sqrt{2}} = 0 \quad , \quad (5.78)$$

and the corresponding Tresca yield surface is the inscribed hexagonal prism defined by the six planes

$$\begin{aligned}\bar{\sigma}_3 - \bar{\sigma}_1 &= \pm 2k, \\ \bar{\sigma}_1 - \bar{\sigma}_2 &= \pm 2k, \\ \bar{\sigma}_2 - \bar{\sigma}_3 &= \pm 2k,\end{aligned}\tag{5.80}$$

where k is a constant.

Now we assume that the body is in a state of *uniaxial stress*, and for definiteness take $\bar{\sigma}_3 \neq 0$, $\bar{\sigma}_1 = 0$ and $\bar{\sigma}_2 = 0$. Using (5.78) we obtain the von Mises yield surface normal from

$$\mathbf{N} = \frac{\partial f}{\partial \sigma};$$

the principal components of \mathbf{N} are

$$N_i = \frac{2\bar{\sigma}_i - \bar{\sigma}_j - \bar{\sigma}_k}{\sqrt{(\bar{\sigma}_1 - \bar{\sigma}_2)^2 + (\bar{\sigma}_2 - \bar{\sigma}_3)^2 + (\bar{\sigma}_3 - \bar{\sigma}_1)^2}}.\tag{5.81}$$

Here i, j, k form a cyclic permutation. For $\bar{\sigma} = (0, 0, \bar{\sigma}_3)$ (5.81) becomes

$$\mathbf{N} = \frac{1}{\sqrt{6}} (-1, -1, 2)\tag{5.82}$$

in normalised form, that is, with $|\mathbf{N}| = 1$.

The two normals to the Tresca yield surface at $\bar{\sigma} = (0, 0, \bar{\sigma}_3)$ are similarly obtained from (5.80), and are

$$\mathbf{M}^1 = \frac{1}{\sqrt{2}} (-1, 0, 1) \quad (5.83)$$

and

$$\mathbf{M}^2 = \frac{1}{\sqrt{2}} (0, -1, 1) \quad (5.84)$$

in normalised form.

If we substitute (5.82) in (5.74) we obtain

$$\frac{\rho_0 (U_L)_M^2}{\ell^2} = d_{33} + \bar{\sigma}_3 - G^{-1} A^2 \quad (5.85)$$

and using (5.83) and (5.84) in (5.74) we obtain

$$\frac{\rho_0 (U_L)_T^2}{\ell^2} = d_{33} + \bar{\sigma}_3 - G_{\alpha\beta}^{-1} A^\alpha A^\beta \quad (5.86)$$

where $(U_L)_M$ and $(U_L)_T$ are, respectively, the speeds of propagation of longitudinal waves in materials with the von Mises and Tresca yield criteria,

$$A = d_{3j} M_j + 2\bar{\sigma}_3 M_3, \quad (5.87)$$

and

$$A^\alpha = d_{3j} M_j^\alpha + 2\bar{\sigma}_3 M_3^\alpha. \quad (5.88)$$

It follows that the difference in the squares of von Mises and Tresca wave speeds is given by

$$\frac{\rho_0}{\ell^2} \left[(U_L)_M^2 - (U_L)_T^2 \right] = G_{\alpha\beta}^{-1} A^\alpha A^\beta - G^{-1} A^2 \quad (5.89)$$

In the case of *transverse waves*, using (5.77) we conclude that

$$(U_T)_M = (U_T)_T \quad (5.90)$$

since there is no plastic influence.

The components of $G_{\alpha\beta}$ are found from

$$\begin{aligned} G_{\alpha\beta} &= h_{\alpha\beta} + \mathcal{L}_{ijkl} M_{ij}^\alpha M_{kl}^\beta \\ &= h_{\alpha\beta} + \sum_{i,j=1}^3 \mathcal{L}_{iijj} M_i^\alpha M_j^\beta \\ &= h_{\alpha\beta} + \sum_{i,j=1}^3 (d_{ij} + 2\bar{\sigma}_i \delta_{ij}) M_i^\alpha M_j^\beta \end{aligned} \quad (5.91)$$

in view of (5.63), and (5.91) becomes

$$G = h + \frac{1}{6} (d_{11} + 2(d_{12} - 2d_{13} - 2d_{23}) + d_{22} + 4d_{33} + 8\bar{\sigma}_3) \quad (5.92)$$

for the von Mises yield criterion and

$$\begin{aligned}
 G_{\alpha\beta} = & h_{\alpha\beta} + d_{11} \mathbb{M}_1^{\alpha\beta} + d_{22} \mathbb{M}_2^{\alpha\beta} \\
 & + (d_{33} + 2\bar{\sigma}_3) \mathbb{M}_3^{\alpha\beta} + d_{12} (\mathbb{M}_1^{\alpha\beta} + \mathbb{M}_2^{\alpha\beta}) \\
 & + d_{13} (\mathbb{M}_1^{\alpha\beta} + \mathbb{M}_3^{\alpha\beta}) + d_{23} (\mathbb{M}_2^{\alpha\beta} + \mathbb{M}_3^{\alpha\beta})
 \end{aligned} \tag{5.93}$$

for the Tresca criterion, with \mathbb{M}^α given by (5.83) and (5.84).

For the given state of stress in isotropic materials we have the condition

$$c_1^e = c_2^e, \quad ,$$

and we also have $B_{1111} = B_{2222}$ and $B_{1133} = B_{2233}$, so that

$$d_{11} = d_{22} \text{ and } d_{13} = d_{23}$$

from (5.60). Thus (5.92) and (5.93) reduce to

$$G = h + \frac{1}{3} (d_{11} + 2d_{33} + d_{12} - 4d_{13} + 4\bar{\sigma}_3) \quad , \tag{5.94}$$

and

$$\begin{aligned}
 G_{\alpha\beta} = & h_{\alpha\beta} + d_{11} (\mathbb{M}_1^{\alpha\beta} + \mathbb{M}_2^{\alpha\beta}) \\
 & + d_{13} (\mathbb{M}_1^{\alpha\beta} + \mathbb{M}_2^{\alpha\beta} + \mathbb{M}_3^{\alpha\beta} + \mathbb{M}_3^{\alpha\beta}) \\
 & + d_{12} (\mathbb{M}_1^{\alpha\beta} + \mathbb{M}_2^{\alpha\beta}) + (d_{33} + 2\bar{\sigma}_3) \mathbb{M}_3^{\alpha\beta} .
 \end{aligned} \tag{5.95}$$

If we substitute (5.83) and (5.84) in (5.95) we obtain the elements

$$G_{11} = h_{11} + \frac{1}{2} (d_{11} - 2d_{13} + d_{33} + 2\bar{\sigma}_3) ,$$

$$G_{12} = G_{21} = h_{12} + \frac{1}{2} (-2d_{13} + d_{12} + d_{33} + 2\bar{\sigma}_3) ,$$

and

$$G_{22} = h_{22} + \frac{1}{2} (d_{11} - 2d_{13} + d_{33} + 2\bar{\sigma}_3)$$

of $G_{\alpha\beta}$.

If we define

$$\xi = \frac{1}{2} (d_{11} - 2d_{13} + d_{33} + 2\bar{\sigma}_3) \quad (5.96)$$

and

$$\eta = \frac{1}{2} (d_{12} - 2d_{13} + d_{33} + 2\bar{\sigma}_3) = \xi + \frac{1}{2} (d_{12} - d_{11}) \quad (5.97)$$

then we can express (5.95) in the matrix form

$$[G] = [h] + \begin{bmatrix} \xi & \eta \\ \eta & \xi \end{bmatrix}$$

or, recalling (3.132),

$$[G] = \begin{bmatrix} h + \xi & h\phi + \eta \\ h\phi + \eta & h + \xi \end{bmatrix} . \quad (5.98)$$

Compressible Neo-Hookean Materials

The free energy function for compressible Neo-Hookean materials [16] can be obtained from (3.89) by assuming $b = 0$ and $e = 0$, to give

$$\psi_0(\mathbf{F}^e) = a|\mathbf{F}^e|^2 + c(\det \mathbf{C}^e) - d \log(\det \mathbf{C}^e)^{\frac{1}{2}}$$

or

$$\psi_0(\mathbf{C}^e) = a \operatorname{tr} \mathbf{C}^e + c \det \mathbf{C}^e - \frac{d}{2} \log(\det \mathbf{C}^e) \quad . \quad (5.99)$$

In terms of the squares of the elastic principal stretches c_i^e (5.99) becomes

$$\psi_0(c_1^e, c_2^e, c_3^e) = a(c_1^e + c_2^e + c_3^e) + c(c_1^e c_2^e c_3^e) - \frac{d}{2} \log(c_1^e c_2^e c_3^e) \quad . \quad (5.100)$$

Recalling that $c_1^e = c_2^e$ and using (5.100) in (5.64) we obtain

$$\bar{\sigma}_i(c_1^e, c_2^e, c_3^e) = 2\rho_0(ac_i^e + cJ^{e^2} - \frac{d}{2}) \quad (5.101)$$

where $J^{e^2} = c_1^e c_2^e c_3^e$. We furthermore require that the body be undeformed in the unstressed state; that is, $\bar{\sigma}_i(1,1,1) = 0$. The constants a , c and d thus have to satisfy

$$2a + 2c = d \quad . \quad (5.102)$$

For the uniaxial stress case with $\bar{\sigma}_3 \neq 0$ we obtain

$$\bar{\sigma}_1 = 0 \quad \Rightarrow \quad ac_1^e + cJ^{e^2} = \frac{d}{2}$$

and

$$\bar{\sigma}_2 = 0 \quad \Rightarrow \quad ac_2^e + cJ^{e^2} = \frac{d}{2} \quad ,$$

and these give

$$c_1^e = c_2^e = \frac{-a + \sqrt{a^2 + 2cdc_3^e}}{2cc_3^e} \quad (5.103)$$

for given c_3^e (bearing in mind that we require $c_i^e > 0$). Hence, the use of (5.102) in (5.99) yields

$$\bar{\sigma}_3 = 2\rho_0 a(c_3^e - c_1^e) \quad (5.104)$$

We obtain the components

$$B_{iijj} = 2\rho_0 \frac{\partial^2 \psi}{\partial c_i^e \partial c_j^e} \quad (\text{no sum})$$

of \mathbf{B} , using (5.100) with (3.116); these are

$$B_{1111} = B_{2222} = 2\rho_0 \frac{\partial^2 \psi}{\partial c_1^{e2}} = \frac{\rho_0 d}{(c_1^e)^2} ,$$

$$B_{1122} = 2\rho_0 \frac{\partial^2 \psi}{\partial c_1^e \partial c_2^e} = 2\rho_0 c c_3^e , \quad (5.105)$$

$$B_{1133} = 2\rho_0 \frac{\partial^2 \psi}{\partial c_1^e \partial c_3^e} = 2\rho_0 c c_2^e ,$$

and

$$B_{3333} = 2\rho_0 \frac{\partial^2 \psi}{\partial c_3^{e2}} = \frac{\rho_0 d}{(c_3^e)^2} .$$

If we use results (5.105) in (5.60) we obtain

$$d_{11} = d_{22} = d_{33} = 2\rho_0 d$$

and (5.106)

$$d_{12} = d_{13} = d_{23} = 4\rho_0 c c_1^e c_2^e c_3^e .$$

Using (5.96), (5.101) and (5.106) we obtain

$$\xi = \rho_0 (d - 2c c_1^e c_2^e c_3^e + 2a c_3^e) \quad (5.107)$$

and, from (5.97),

$$\eta = 2\rho_0 a c_3^e . \quad (5.108)$$

Suppose now that we assume that independent hardening takes place with the Tresca criterion, so that $\phi = 0$. Then

$$[G] = \begin{bmatrix} h + \xi & \eta \\ \eta & h + \xi \end{bmatrix} \quad (5.109)$$

from (5.98). The characteristic values λ_1 and λ_2 of $[G]$ are

$$\lambda_1, \lambda_2 = h + \xi \pm \eta \quad (5.110)$$

and the corresponding characteristic vectors in normalised form are

$$p_1 = \frac{1}{\sqrt{2}} (1, 1) \quad (5.111)$$

and

$$p_2 = \frac{1}{\sqrt{2}} (-1, 1) \quad (5.112)$$

We can thus express $[G]$ in the form

$$[G] = [S] [\Lambda] [S]^{-1} \quad (5.113)$$

where

$$[\Lambda] = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$$

is the characteristic value matrix, and

$$[S] = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \quad (5.114)$$

is the characteristic vector matrix, which has the property

$[S]^T = [S]^{-1}$. It follows that

$$\begin{aligned} G_{\alpha\beta}^{-1} A^\alpha A^\beta &= [A]^T [G]^{-1} [A] = [B]^T [A]^{-1} [B] \\ &= \sum_{i=1}^2 \lambda_i^{-1} B_i^2 \end{aligned} \quad (5.115)$$

where

$$[B] = [S]^T [A] \quad (5.116)$$

From (5.82) and (5.87),

$$A = \frac{2}{\sqrt{6}} (-d_{31} + d_{33} + 2\bar{\sigma}_3) \quad (5.117)$$

for the von Mises yield surface.

Since

$$[B] = \frac{1}{\sqrt{2}} \begin{bmatrix} A^1 + A^2 \\ -A^1 + A^2 \end{bmatrix} \quad (5.118)$$

we find from (5.115) that

$$G_{\alpha\beta}^{-1} A^\alpha A^\beta = \frac{1}{2} \left[\lambda_1^{-1} (A^1 + A^2) + \lambda_2^{-1} (A^1 - A^2)^2 \right] \quad (5.119)$$

where

$$A^1 = d_{3j} \mathbb{M}_j^1 + 2\bar{\sigma}_3 \mathbb{M}_j^1 = \frac{1}{\sqrt{2}} (-d_{31} + d_{33} + 2\bar{\sigma}_3) \quad (5.120)$$

and

$$A^2 = d_{3j} \mathbb{M}_j^2 + 2\bar{\sigma}_3 \mathbb{M}_j^2 = \frac{1}{\sqrt{2}} (-d_{31} + d_{33} + 2\bar{\sigma}_3) \quad (5.121)$$

Thus

$$A^1 = A^2 = \frac{\sqrt{3}}{2} A \quad (5.122)$$

If we substitute (5.122) into (5.118) we obtain

$$[B] = \frac{\sqrt{3}}{2} \begin{bmatrix} A \\ 0 \end{bmatrix} \quad (5.123)$$

Returning now to (5.89) we find that

$$G_{\alpha\beta}^{-1} A^\alpha A^\beta - G^{-1} A^2 = \frac{3}{2} (h + \xi - \eta)^{-1} - G^{-1} \quad (5.124)$$

where, from (5.94) and (5.106),

$$G = h + \frac{3}{2} \rho_0 (d + 4ac_3^e - J e^2) \quad (5.125)$$

With the expression $J^{e^2} = \frac{1}{2} \frac{d}{c} - \frac{a}{c} c_1^e$ we have

$$G = h + \frac{4}{3} \rho_0 a (2c_3^e - c_1^e) \quad . \quad (5.126)$$

From (5.97) and (5.106),

$$h + \xi - \eta = h + 2\rho_0 a c_1^e \quad . \quad (5.127)$$

Now (5.124) becomes, with $H = \frac{2}{3} (h + \xi - \eta)$,

$$G_{\alpha\beta}^{-1} A^{\alpha\beta} - G^{-1} = A^2 [H^{-1} - G^{-1}] = A^2 H^{-1} G^{-1} (G - H) \quad ,$$

and since $G > 0$ and $H > 0$ (the latter follows from (5.127)), the difference between the speeds of propagation of acceleration waves in von Mises and Tresca materials depends on the sign of $G - H$. In particular

$$G - H \gtrless 0 \Rightarrow (U_L)_T \lessgtr (U_L)_M \quad . \quad (5.128)$$

Equations (5.126) and (5.127) immediately yield

$$(U_L)_T \lessgtr (U_L)_M \quad \text{if } h + 8\rho_0 a (c_3^e - c_1^e) \gtrless 0 \quad (5.129)$$

or, equivalently, if $h + 4 \bar{\sigma}_3 \gtrless 0$.

Given the complexity of this problem, (5.129) is a remarkably simple criterion for determining the relative magnitudes of von Mises and Tresca wave speeds. Certainly we can conclude that $(U_L)_T < (U_L)_M$ for the case of uniaxial tension ($\bar{\sigma}_3 > 0$) and for uniaxial compression with $\bar{\sigma}_3 > -\frac{h}{4}$. Otherwise, $(U_L)_T \geq (U_L)_M$ if $\bar{\sigma}_3 \leq -\frac{h}{4}$.

CHAPTER VI

CONCLUSIONS

We have constructed a finite-strain theory of plasticity which conforms to the standard requirements for such a theory: it has a sound mathematical and physical basis, and it is simple in structure, hence easy to use. It remains to carry out further detailed studies of the range of specific constitutive laws which are contained in the theory presented here, as special cases. This would amount to the following exercise: given a set of constitutive relations in classical form, define the set of internal variables, the free energy function and one of the yield function, dissipation function or maximal responsive map which result in the given set of relations being reproduced exactly by our theory. Naturally this will not always be possible; for example, any theory involving a non-convex potential falls outside the scope of our theory. Nevertheless, it is our contention that the theory of finite-strain plasticity presented in this thesis provides a framework sufficiently broad in which to model a wide range of materials which are elastic-plastic in nature. As in the small-strain case [26], the theory contains in a natural way the three equivalent alternative descriptions of the flow law: that is, using a yield function, or a dissipation function, or finally, by using what is known in the classical literature as the maximum plastic work inequality.

Our theory of finite-strain plasticity is broad enough to cover convex but non-smooth yield surfaces, hence this broadness made it easy to evolve the theory to treat singularities, that is, vertices of the yield surface in a fashion similar to that of Sewell [70,71].

The problem of acceleration wave propagation in elastic-plastic solids with isotropic hardening and convex, but non-smooth yield surfaces which undergo finite strains, has been reduced to the form of a characteristic value problem. The propagation condition and a symmetric, positive-definite acoustic tensor has been obtained. The propagation conditions in the case of elastic-plastic interfaces have been obtained and a deviation from a characteristic value problem has been observed which is similar to the results in [3]. Elastic and plastic wave speeds have been compared by deriving inequalities similar to Mandel's inequalities [47,49] for the case of materials with smooth yield surfaces. Propagation conditions and the expressions for wave speeds have been obtained in the cases of principal, longitudinal and transverse waves. Unlike the case of certain thermoelastic materials with internal constraints [5], it is found in this thesis that the speeds of propagation of longitudinal and transverse principal waves in elastic-plastic solids are equal.

A comparison of the speeds of propagation of acceleration waves in materials with the Tresca and von Mises yield criteria has been made and the influence of yield surface vertices on the speed of propagation has been examined, in the context of a material with a

compressible neo-Hookean elastic law, and in a state of uniaxial stress.

One possible extension of the work presented in this thesis may be the investigation of Mandel-type inequalities for the case of non-smooth yield surfaces. Another outstanding problem concerns the transport (growth and decay) equations of acceleration waves in elastic-plastic solids undergoing finite strains. Finally, another important area which seems to be worthy of investigation is the study of first order discontinuities, that is shock waves in elastic-plastic solids.

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